Seeing plane partitions in Lie representation theory

A different approach to our counting problems

A summary

1. Some general background information

   Algebras, representations, and tensor products

2. $\mathfrak{sl}(2, \mathbb{C})$ and its representation theory

   The action on irreps V_n

3. How PPs appear in tensor products of representations

   "Modified" Kasteleyn-Percus

4. Can we get back to our original KP matrices?

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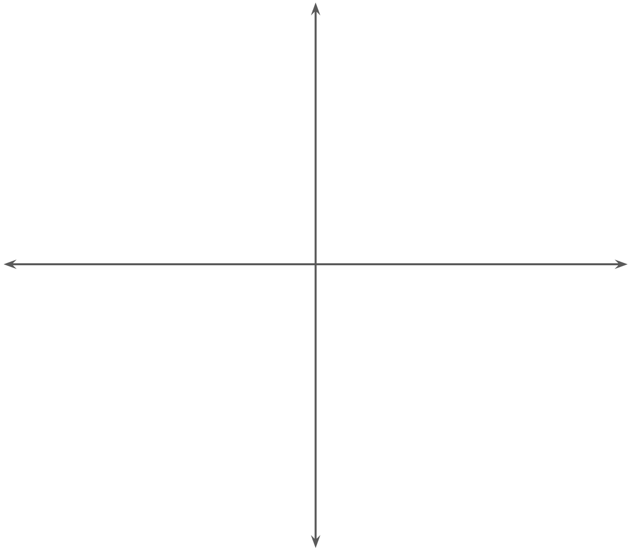

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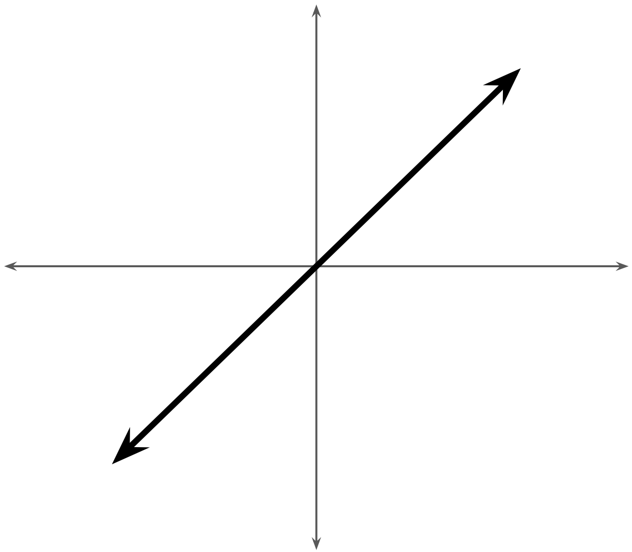

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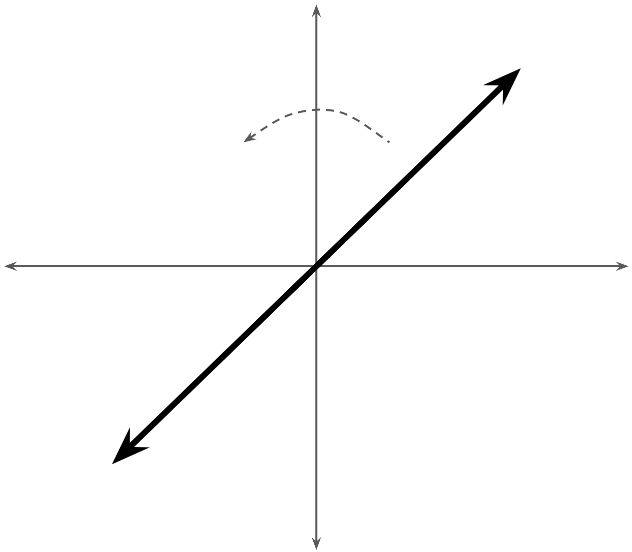

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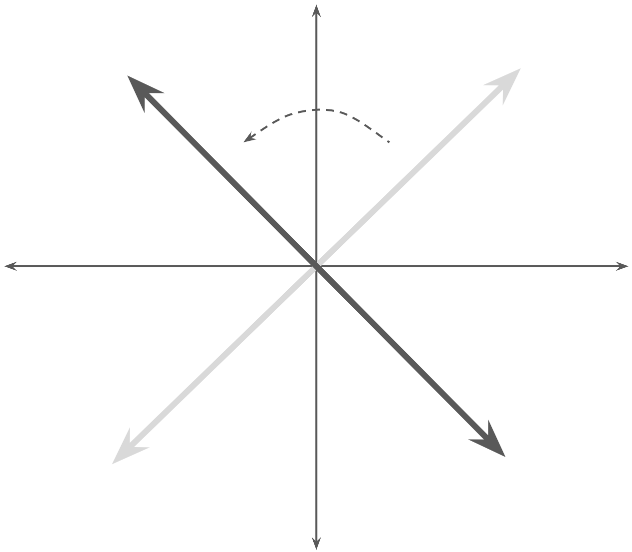

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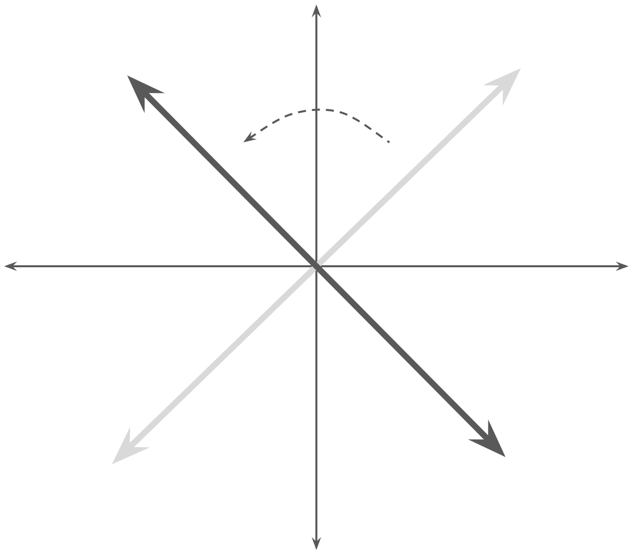

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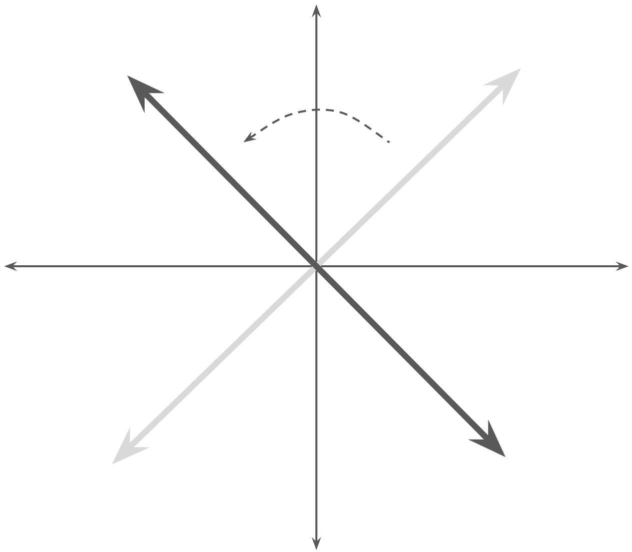

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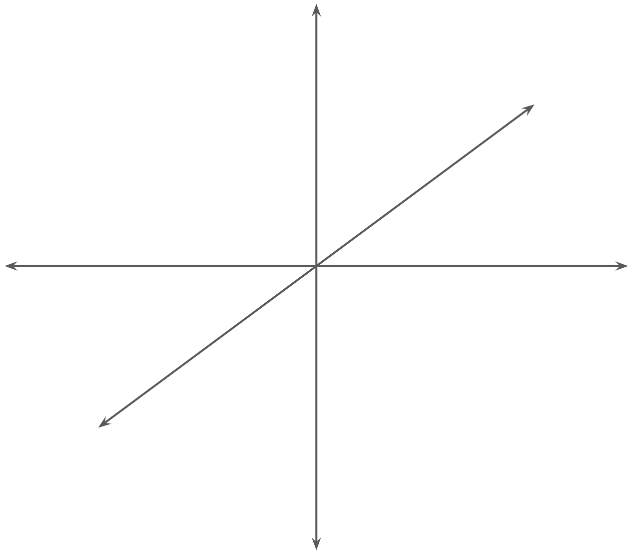

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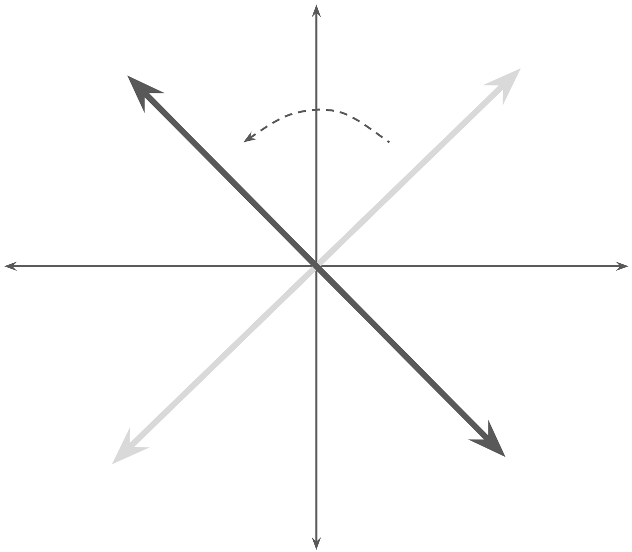

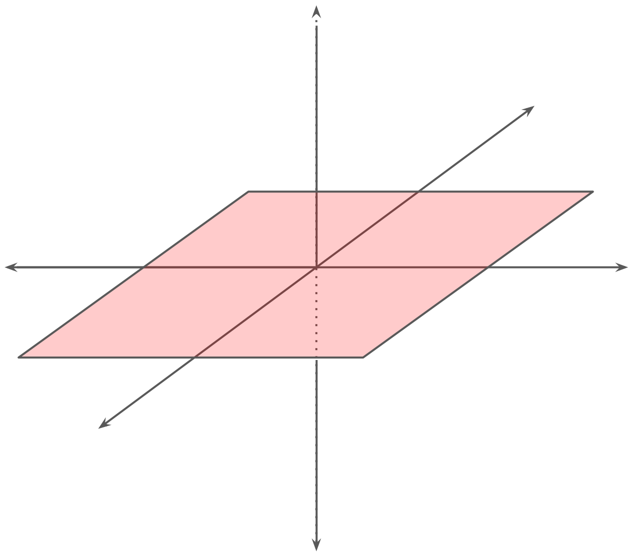

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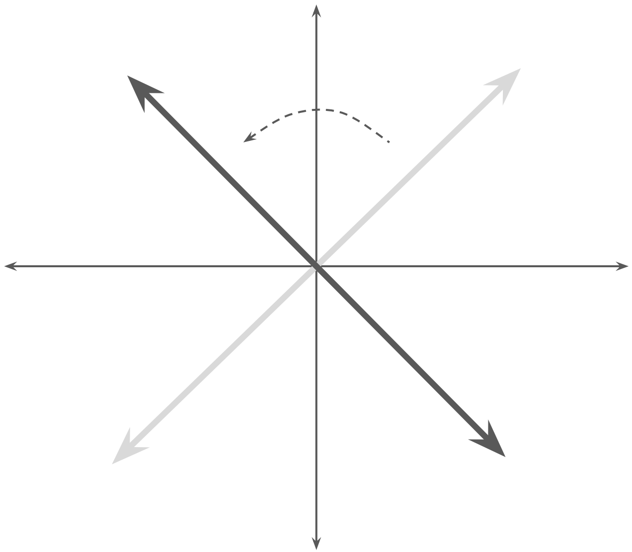

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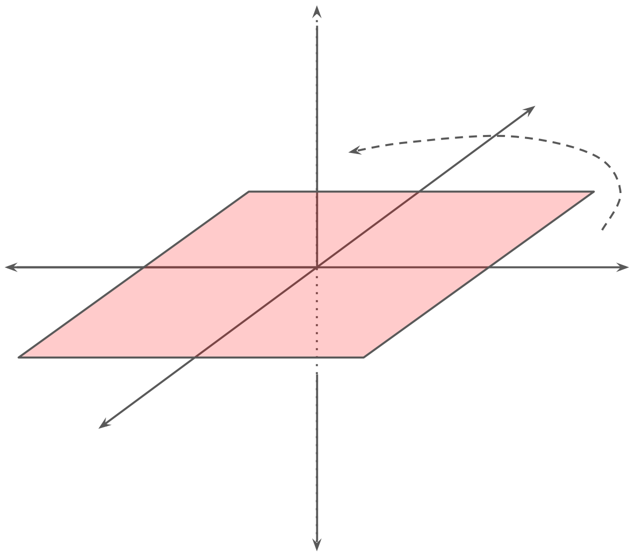

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	$\alpha(X) = x\dfrac{\partial}{\partial y}$	$\alpha(Y) = y\dfrac{\partial}{\partial x}$	$\alpha(H) = x\dfrac{\partial}{\partial x} - y\dfrac{\partial}{\partial y}$
$x^a y^b$	$bx^{a+1}y^{b-1}$	$ax^{a-1}y^{b+1}$	$(a-b)x^a y^b$

What are these V_n?

V_n is the space of homogeneous polynomials of degree *n* in two variables

e.g. V_4 has basis

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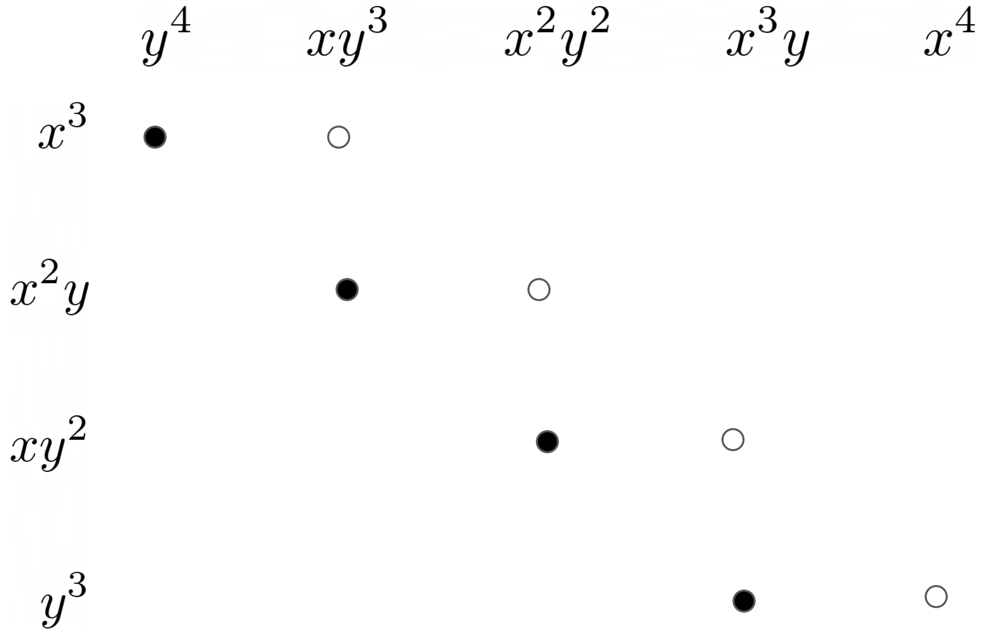

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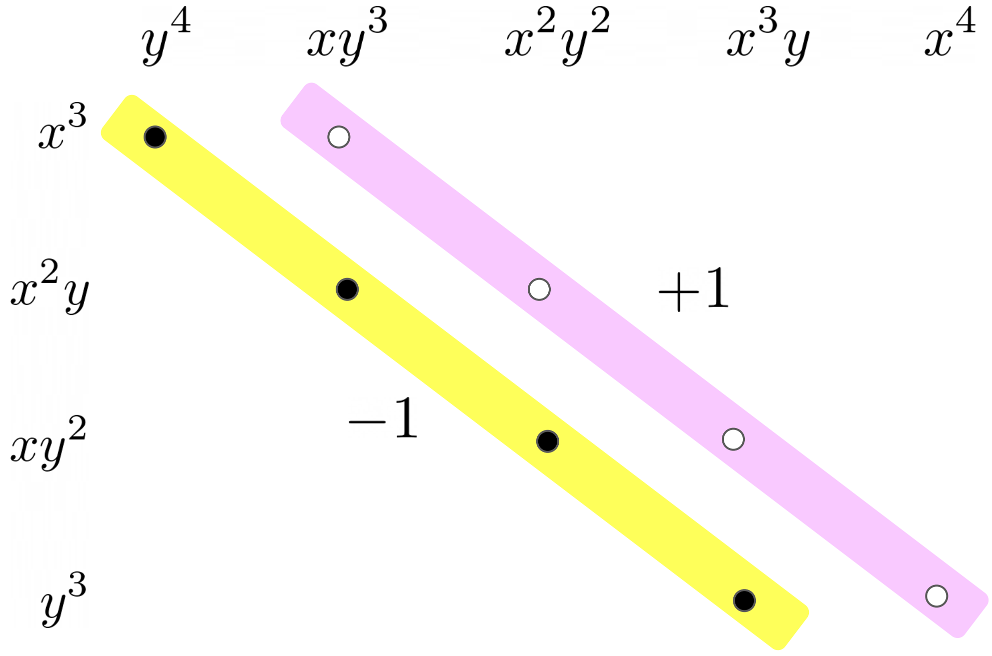

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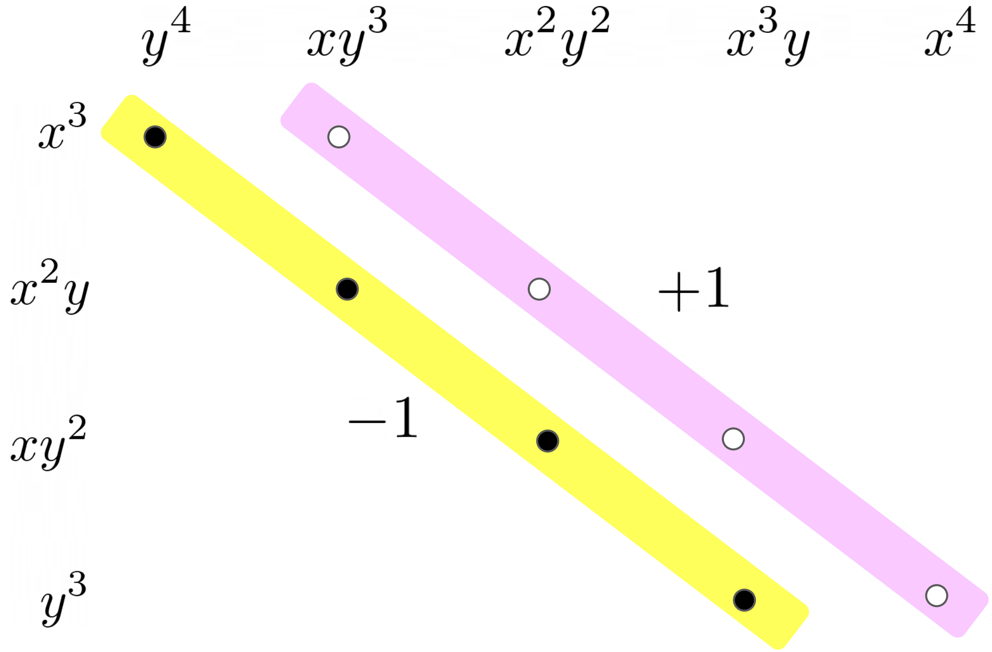

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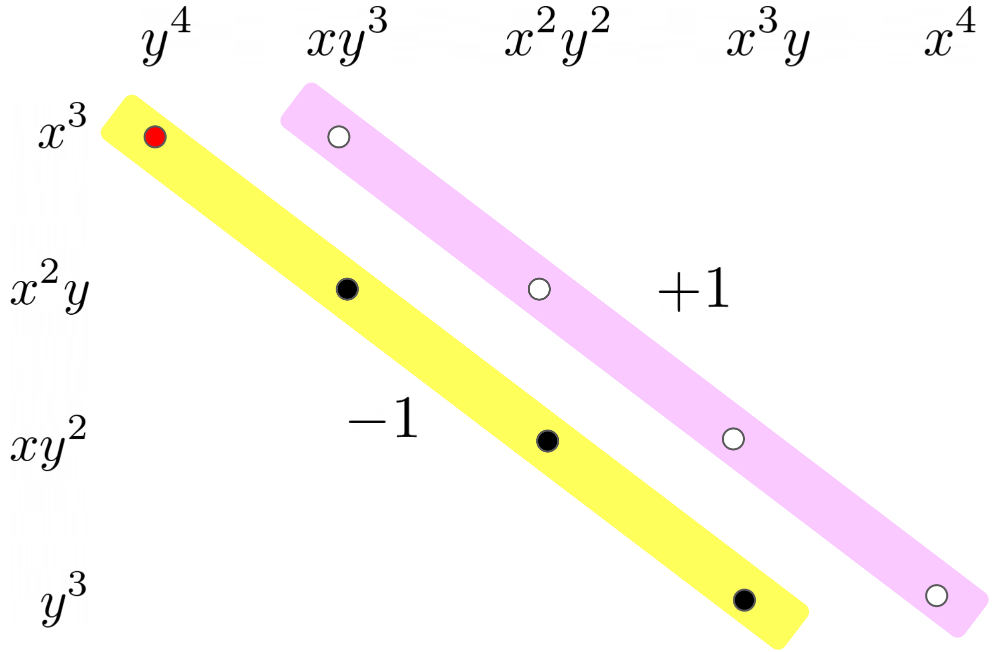

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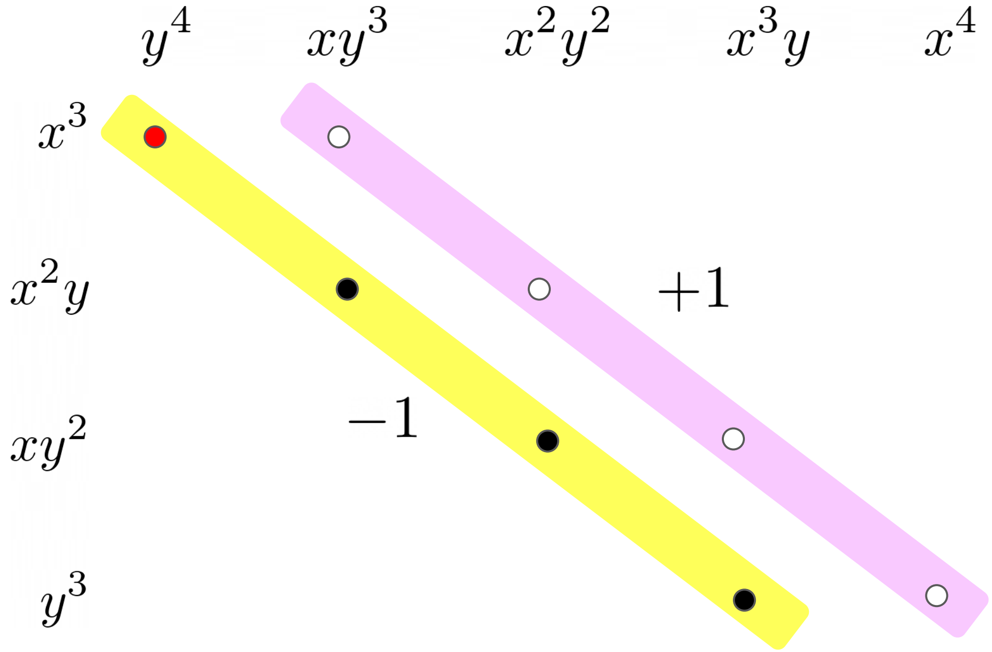

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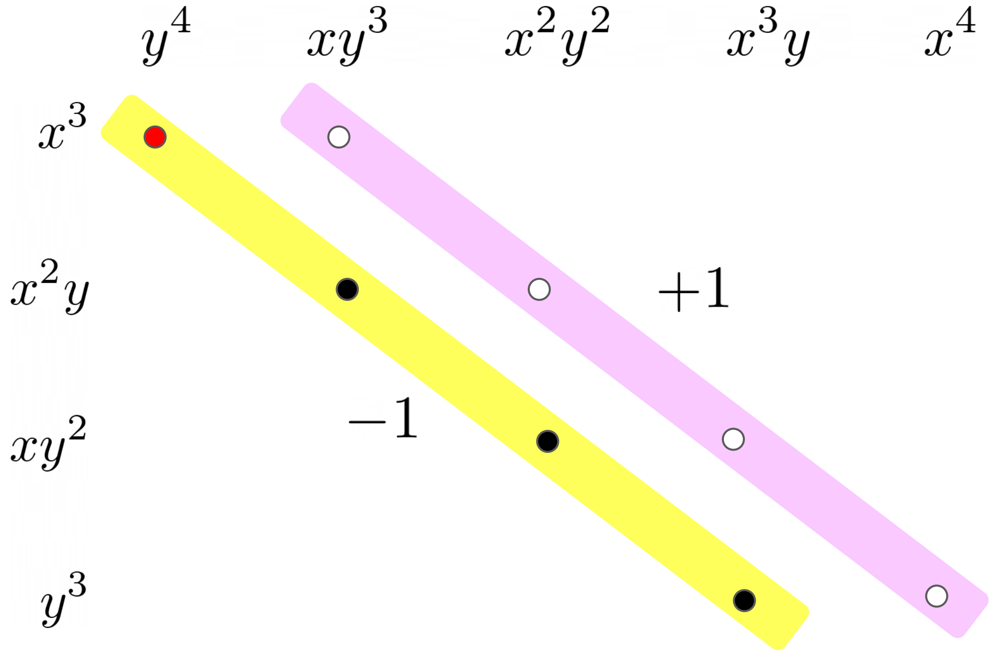

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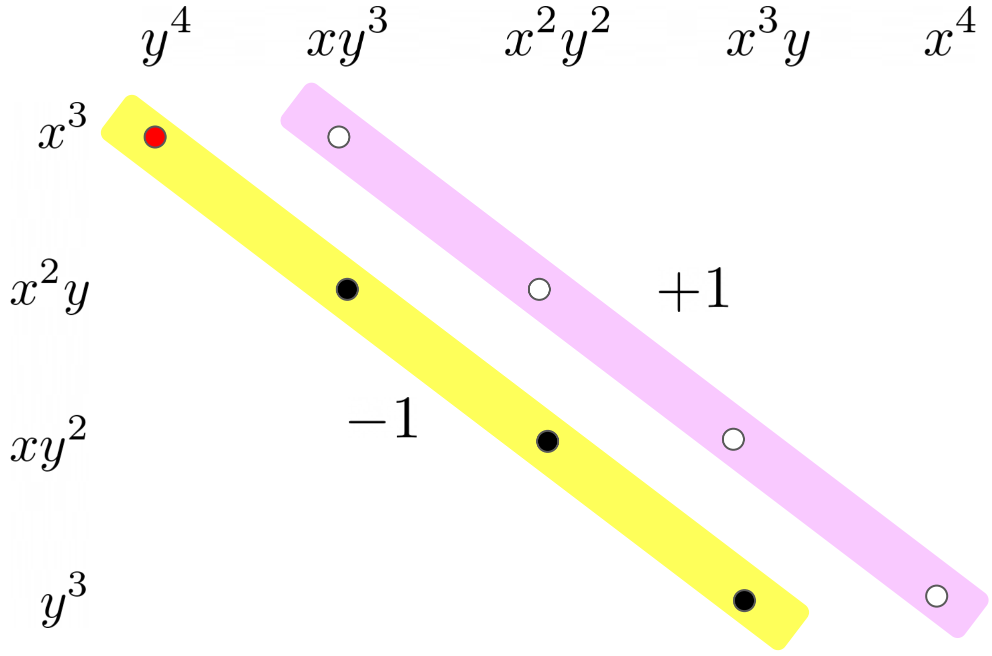

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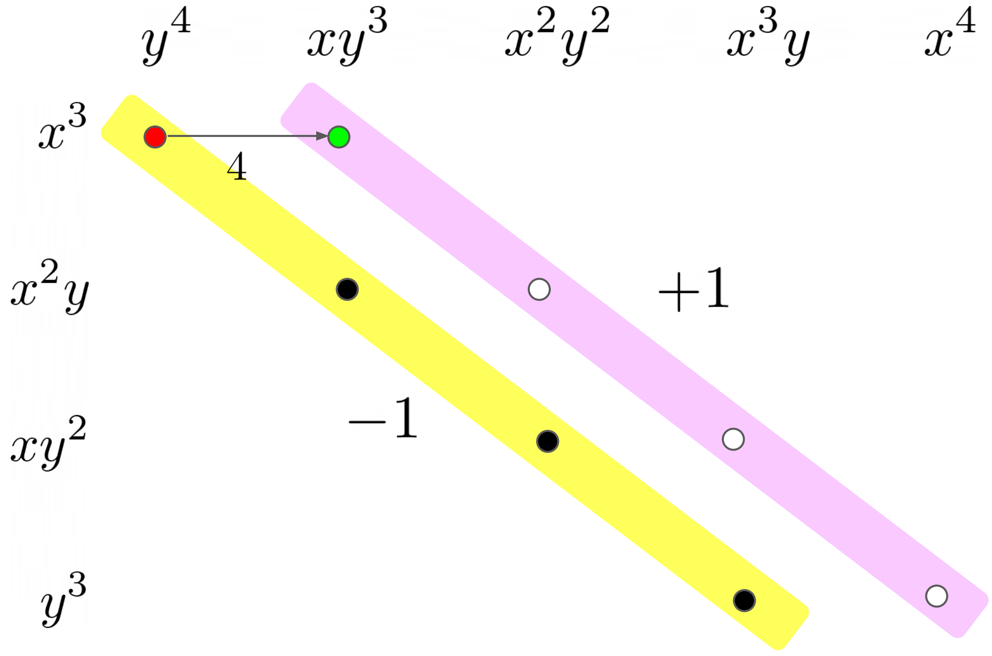

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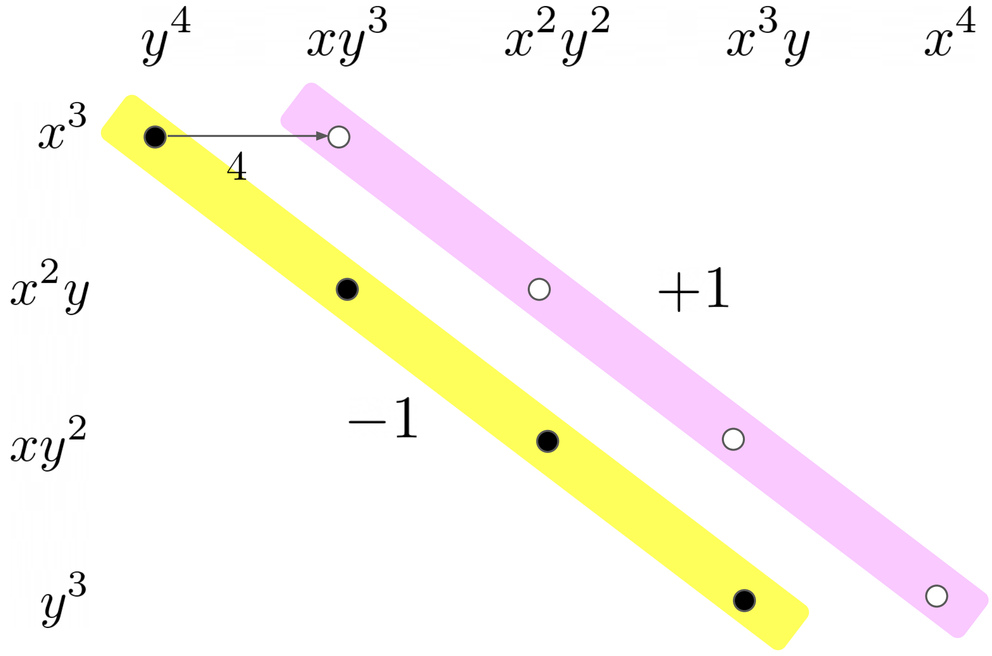

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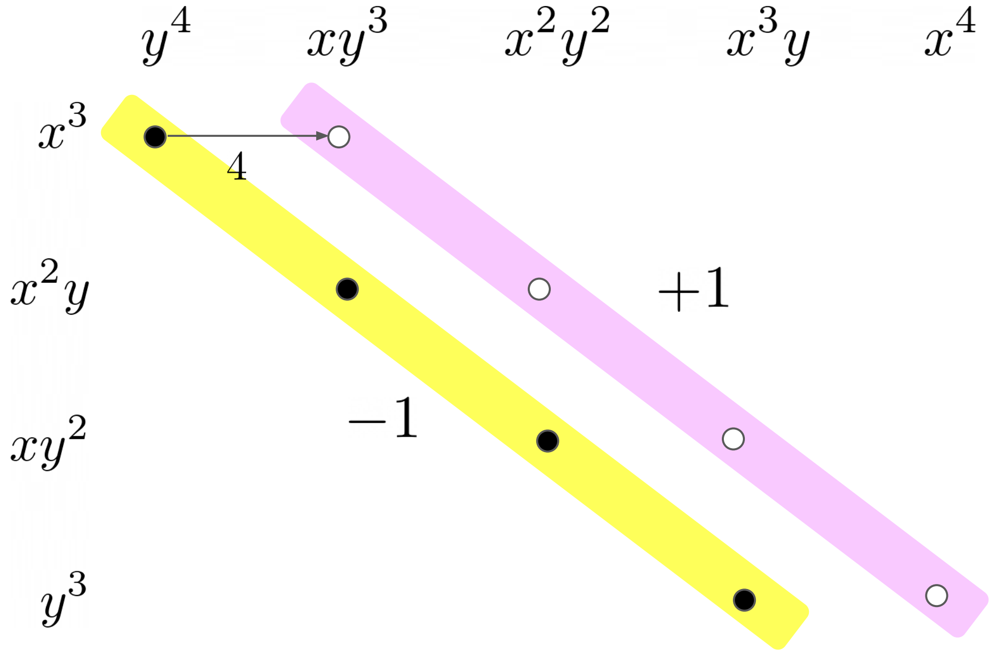

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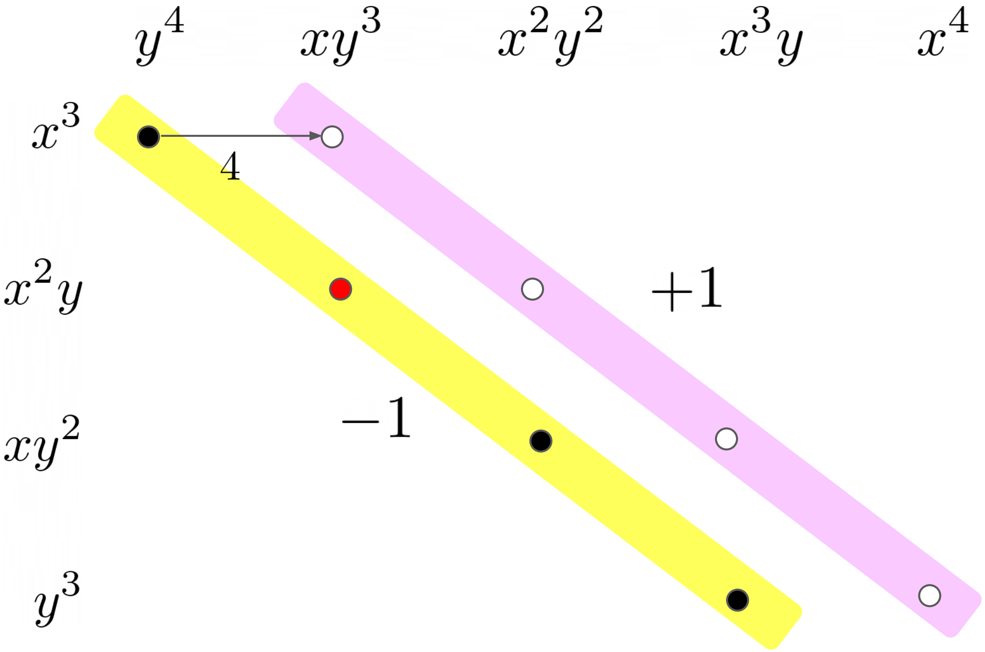

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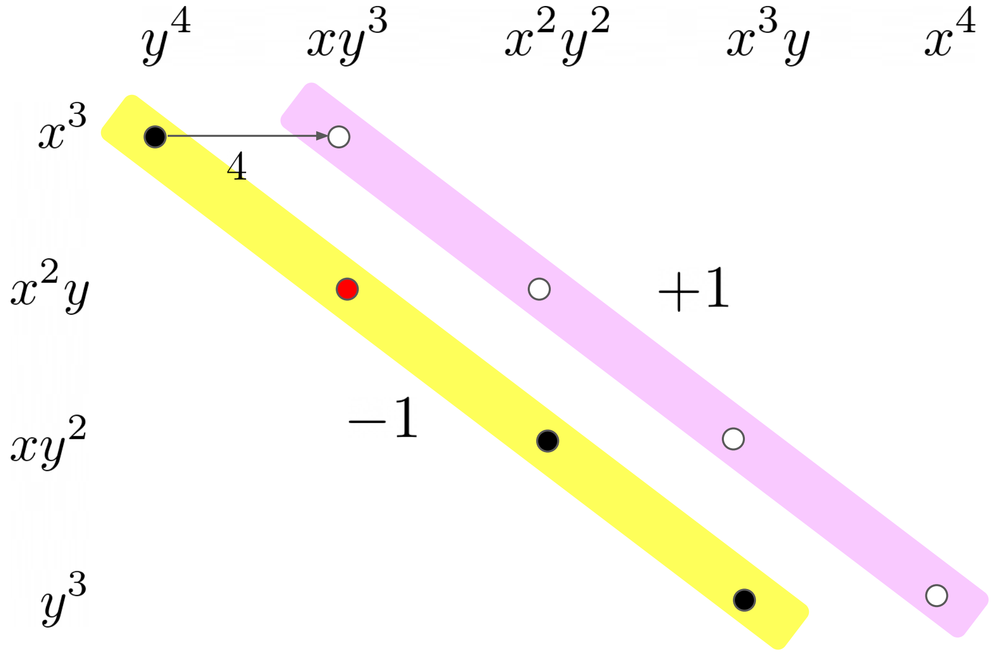

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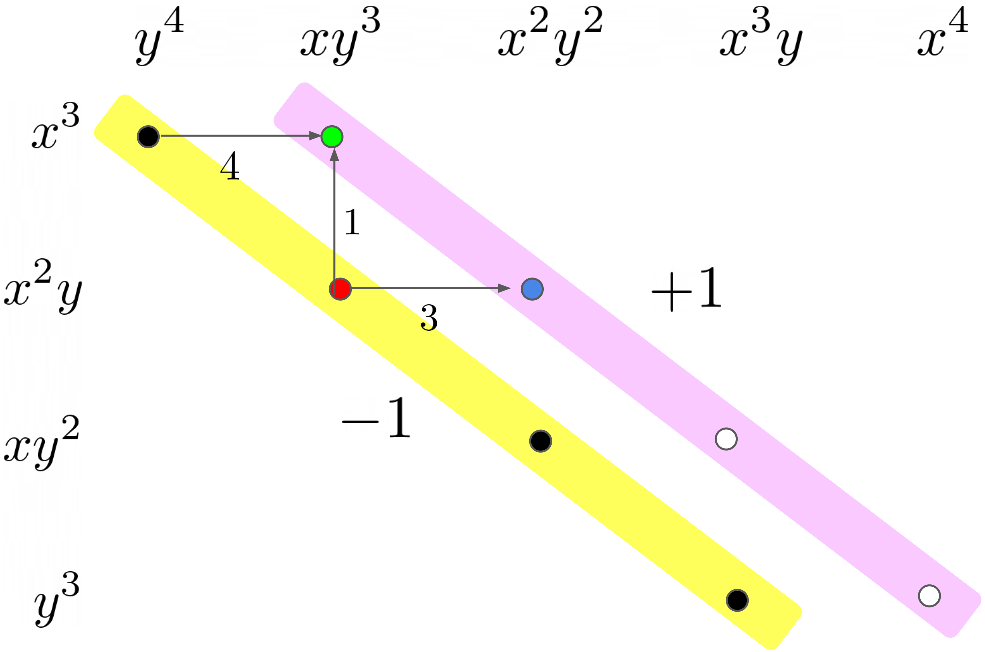

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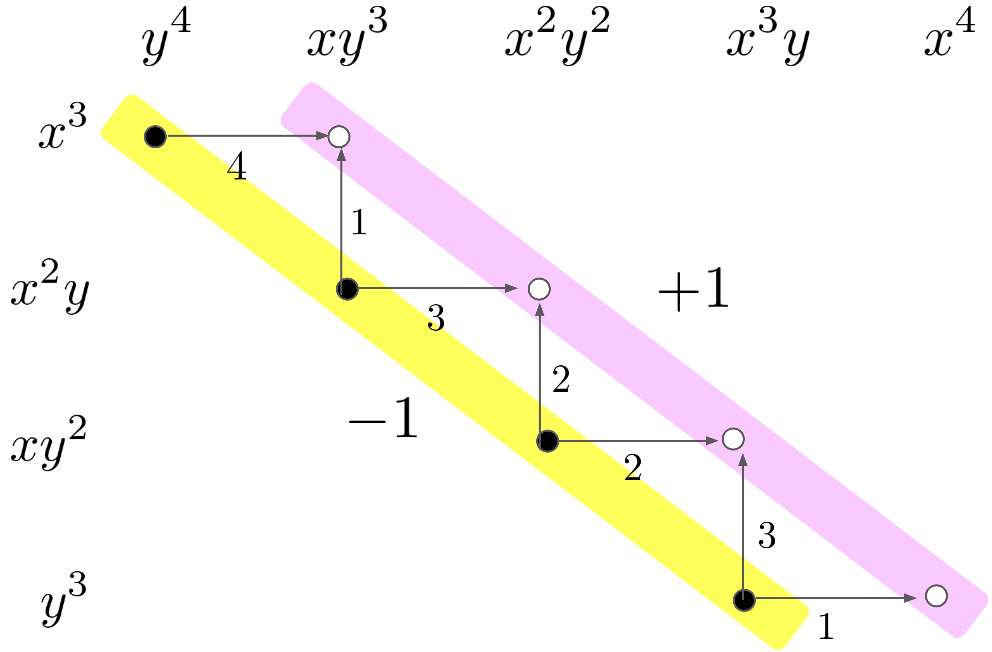

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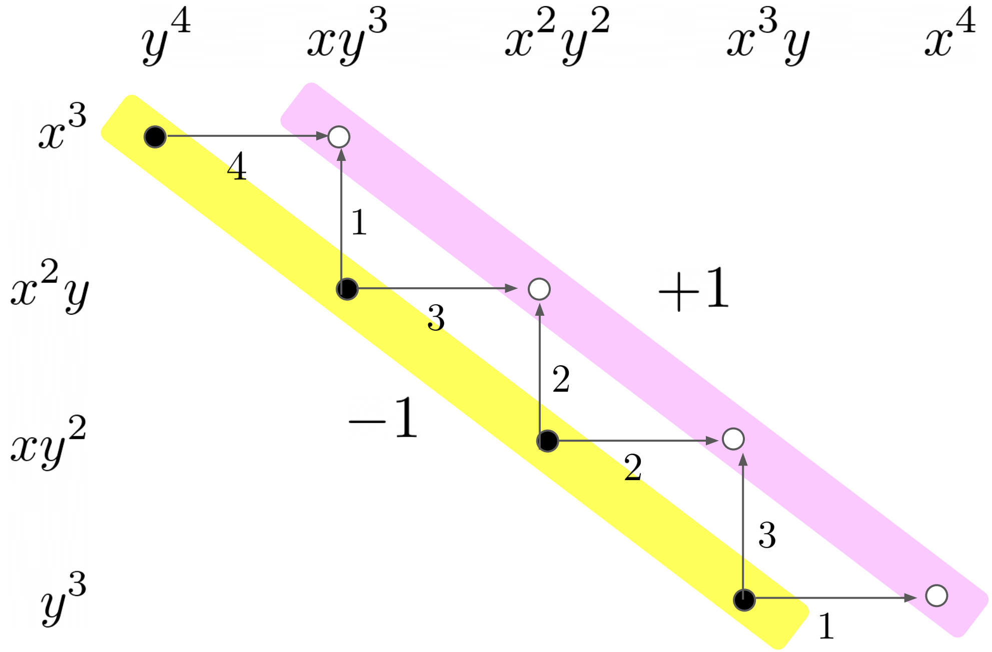

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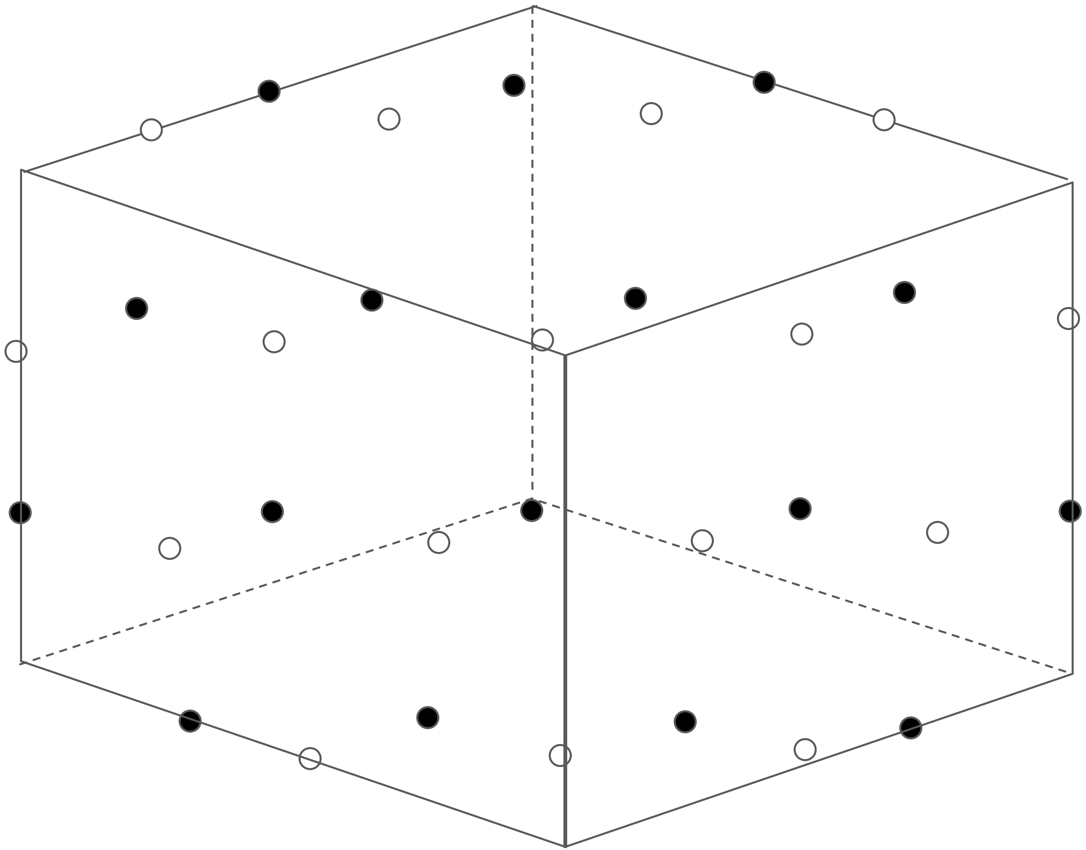

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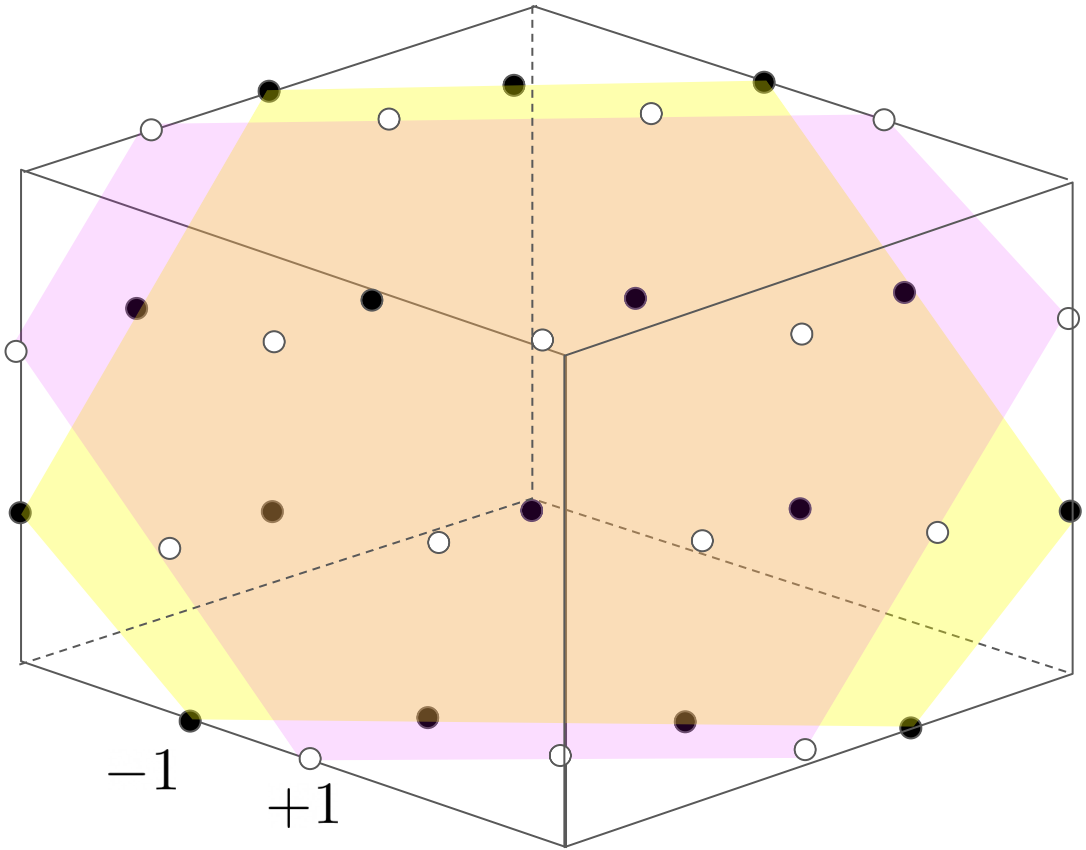

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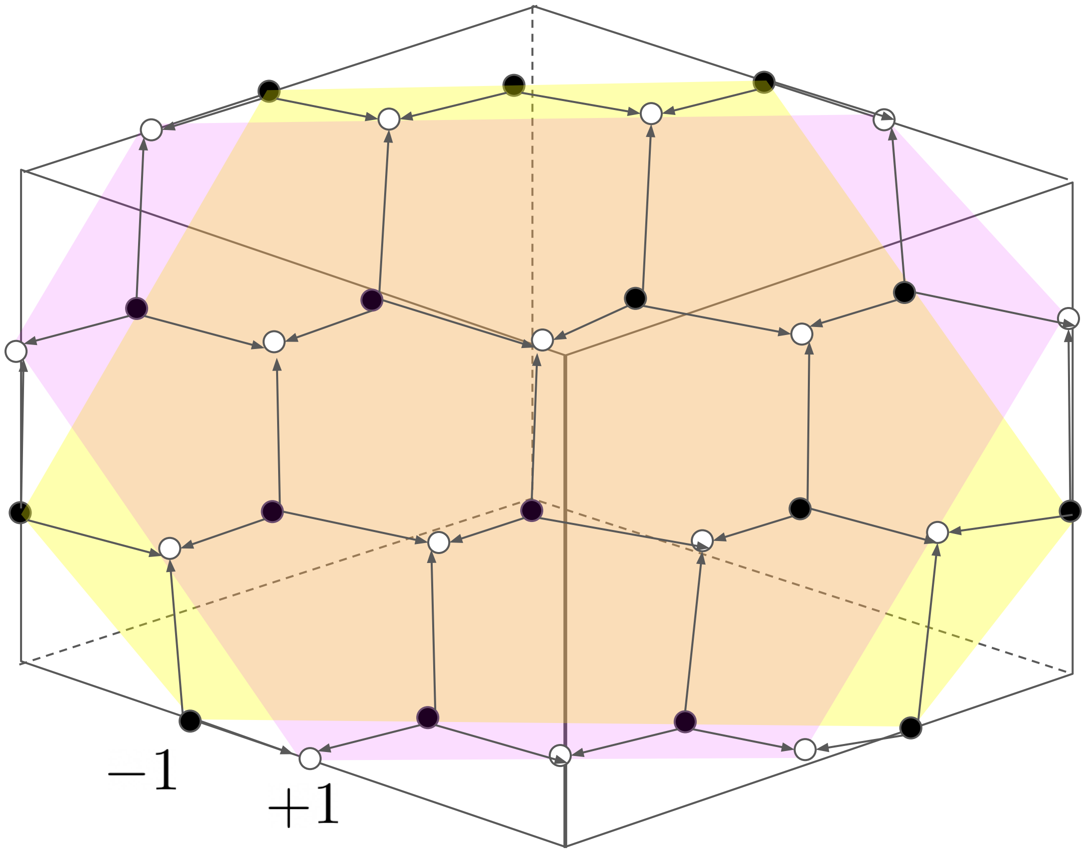

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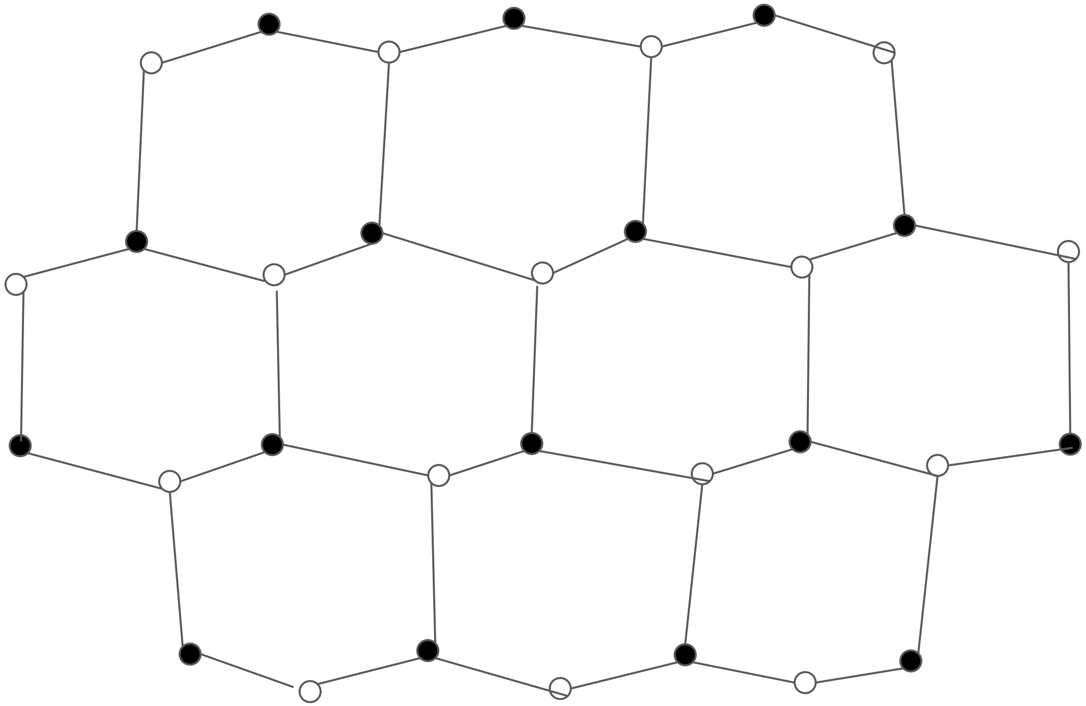

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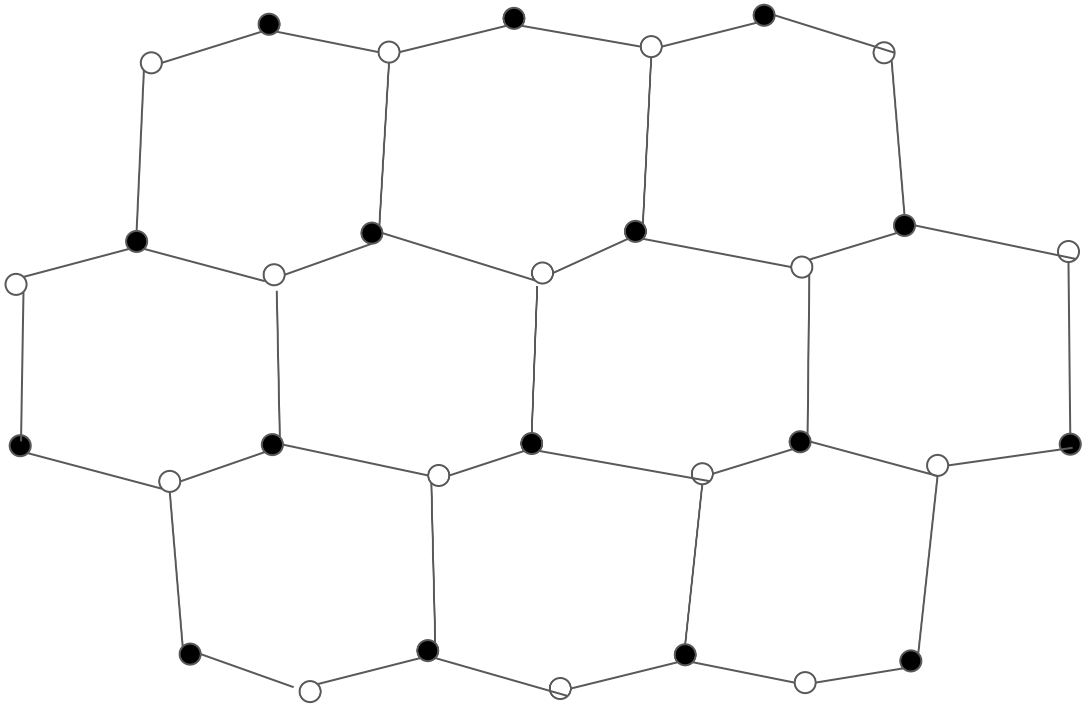

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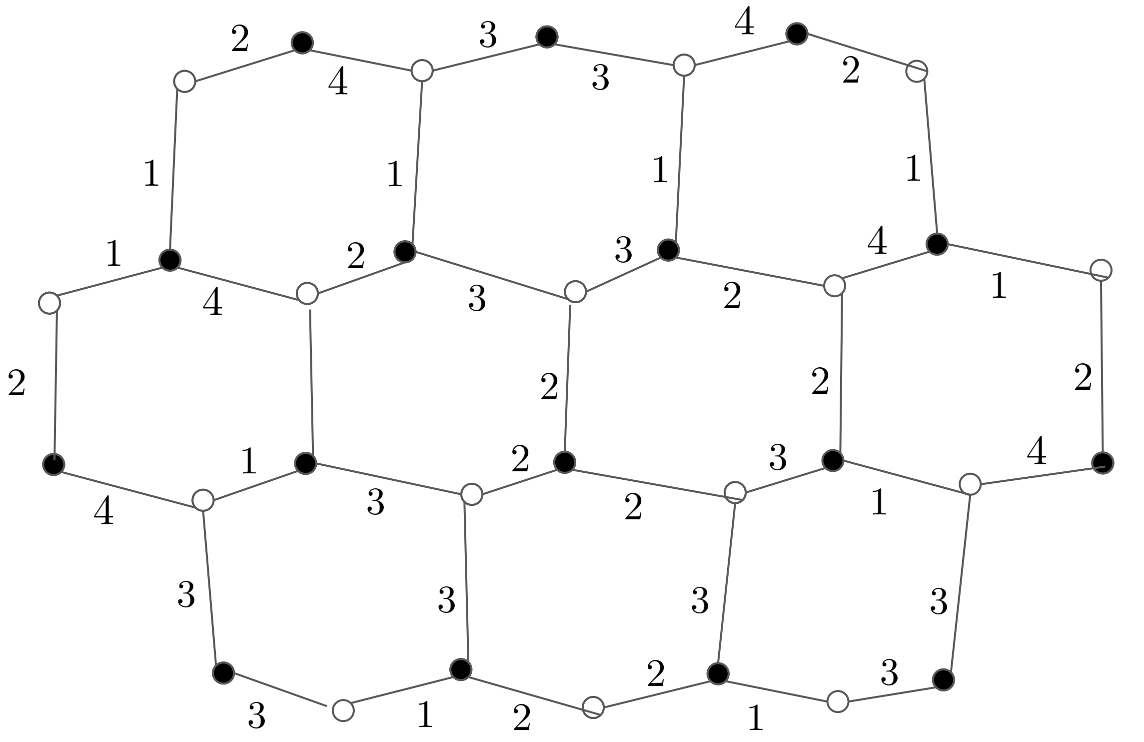

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$\Longrightarrow$ A diagonalized matrix for the action of X

$\Longrightarrow$ A determinant: the solution to our counting problem!

Getting back to our "original" Kasteleyn cokernel

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$2 \times 2 \times 2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/10$	$(\mathbb{Z}/2)^3 \oplus \mathbb{Z}/6 \oplus \mathbb{Z}/12 \oplus \mathbb{Z}/60$
$2 \times 2 \times 3$	$\mathbb{Z}/5 \oplus \mathbb{Z}/10$	$(\mathbb{Z}/2)^3 \oplus (\mathbb{Z}/6)^2 \oplus \mathbb{Z}/12 \oplus (\mathbb{Z}/60)^2$

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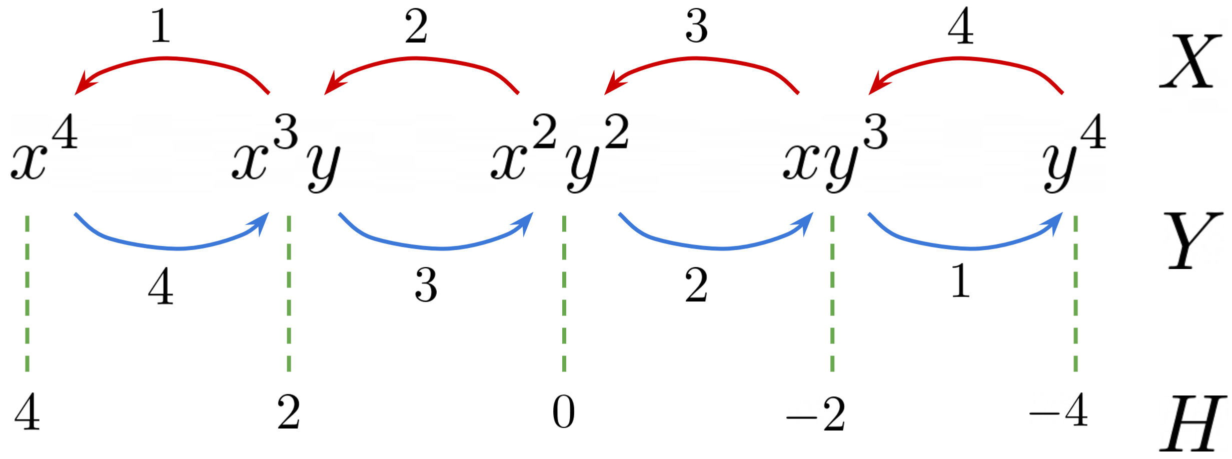

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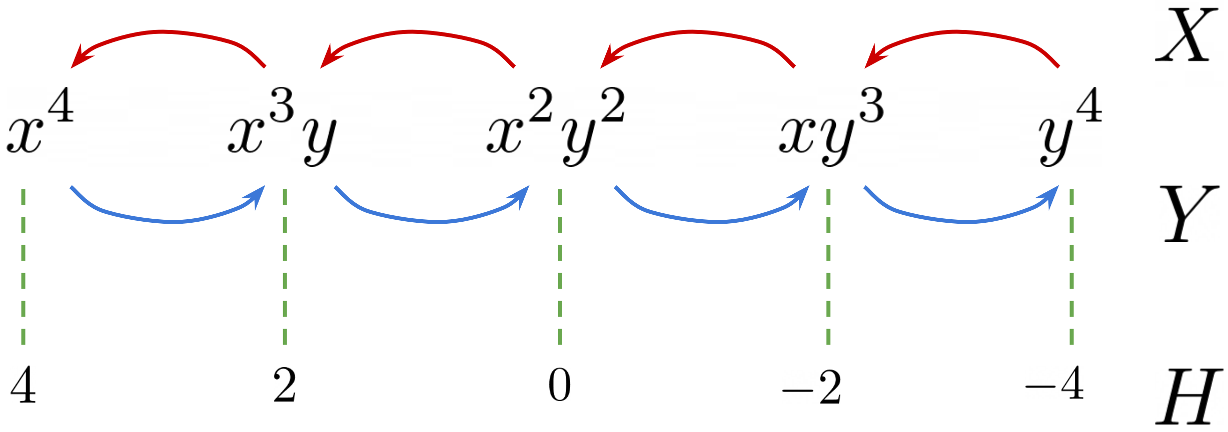

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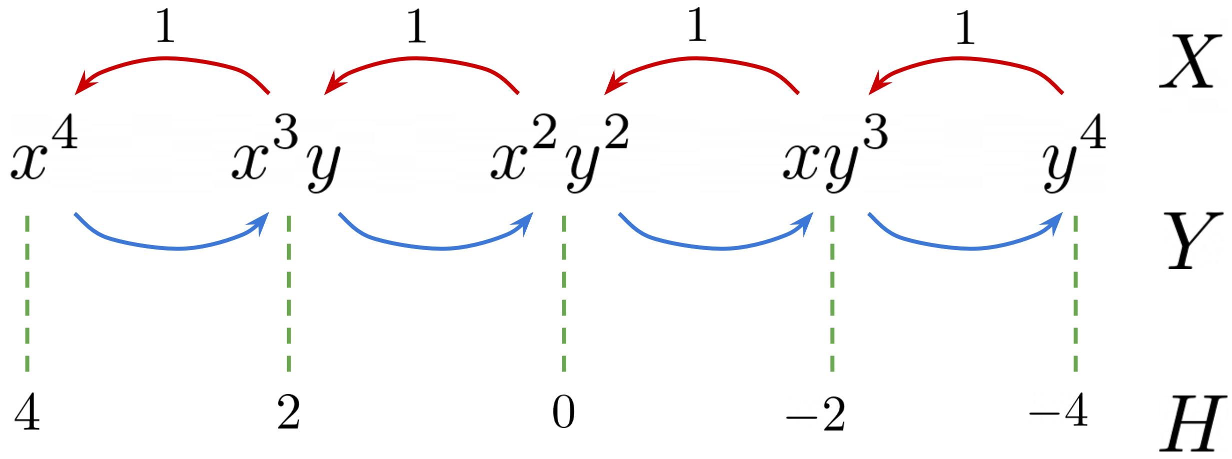

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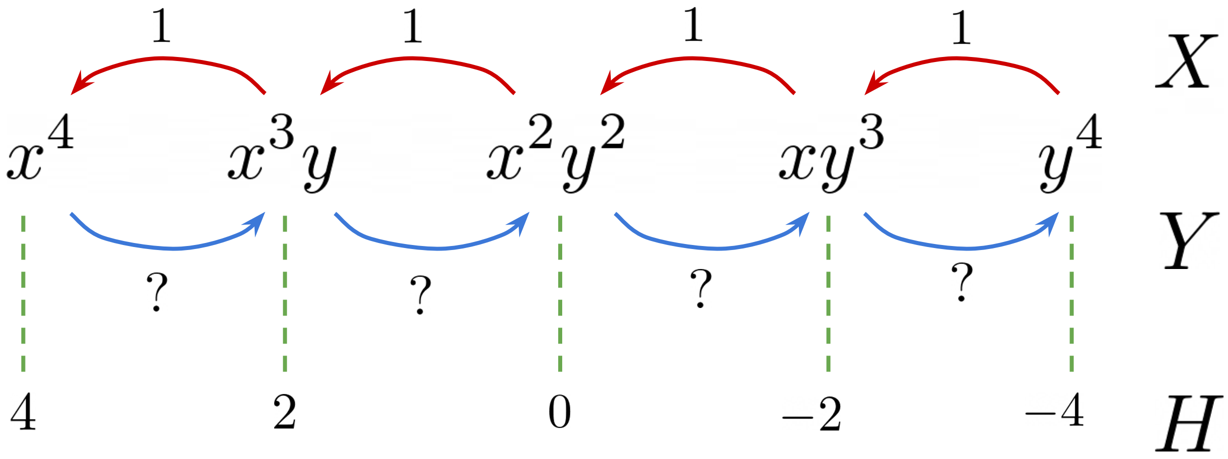

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x^4 \arrow[r, bend right, "?"'] & x^3y \arrow[l, bend right, "1"'] \arrow[r, bend right, "?"'] & x^2y^2 \arrow[l, bend right, "1"'] \arrow[r, bend right, "?"'] & xy^3 \arrow[l, bend right, "1"'] \arrow[r, bend right, "?"'] & y^4 \arrow[l, bend right, "1"'] & \begin{matrix} X \\ Y \end{matrix} \\
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\end{tikzcd}$$

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Another problem...

Our Clebsch-Gordan was over $\mathbb{C}$ ...

But our cokernel needs to be over $\mathbb{Z}$!

Enter ... $\mathfrak{sl}(2, \mathbb{Z})$

Does it have a version of Clebsch-Gordan?

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$$A := \langle a, b + 2c, 4(d + e), 12f \rangle$$

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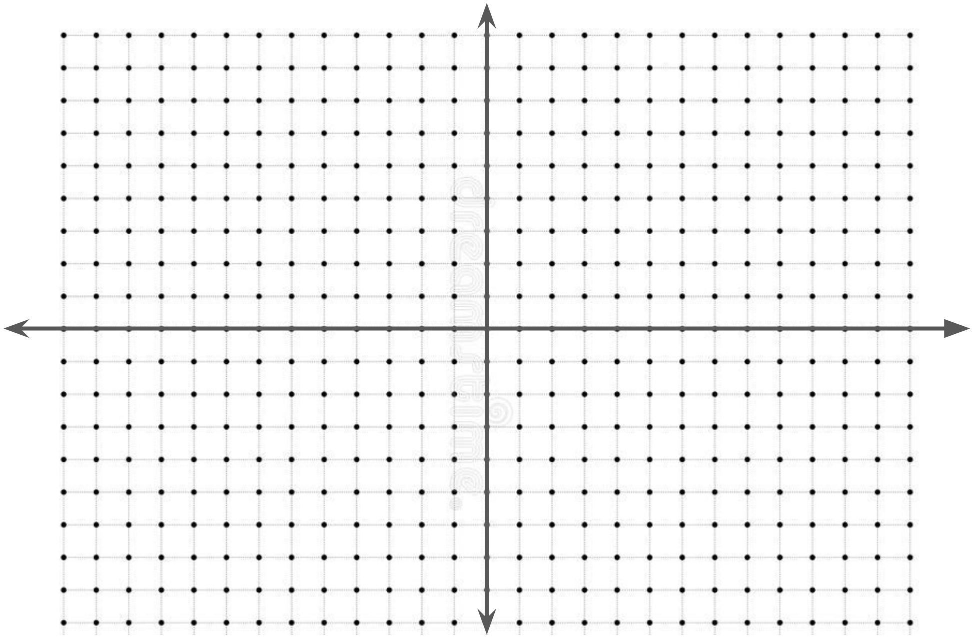

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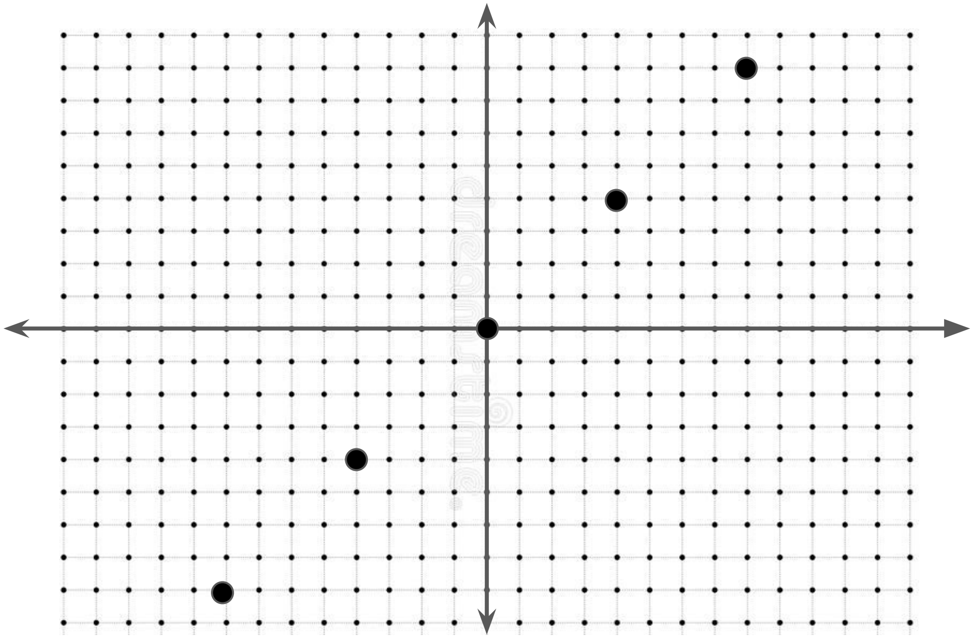

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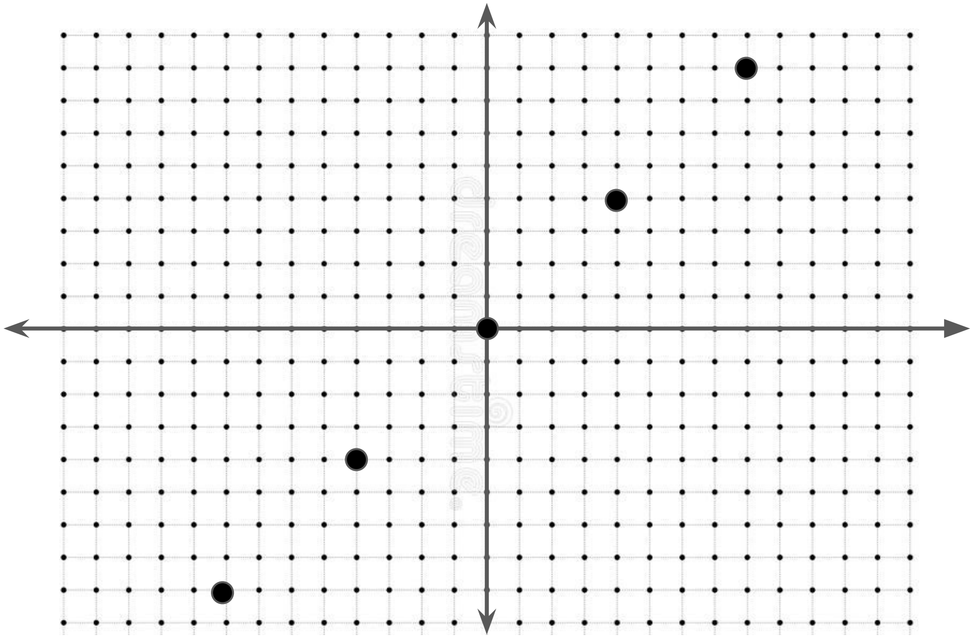

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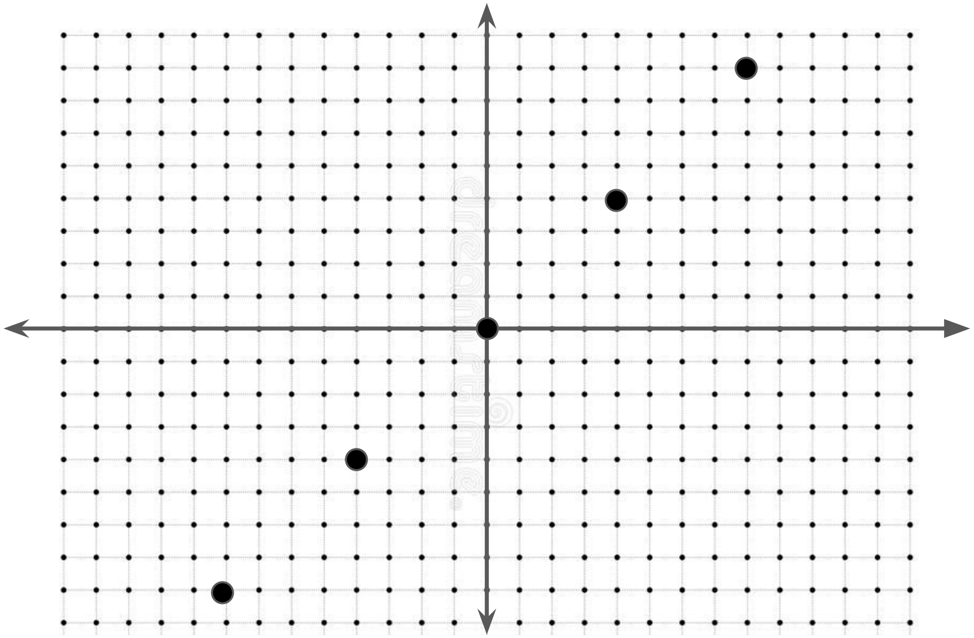

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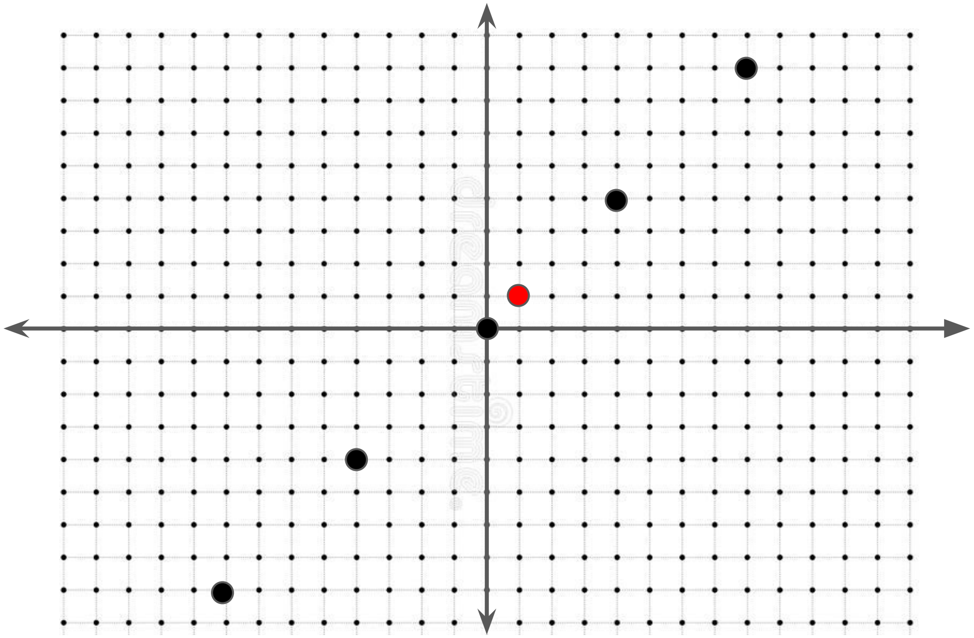

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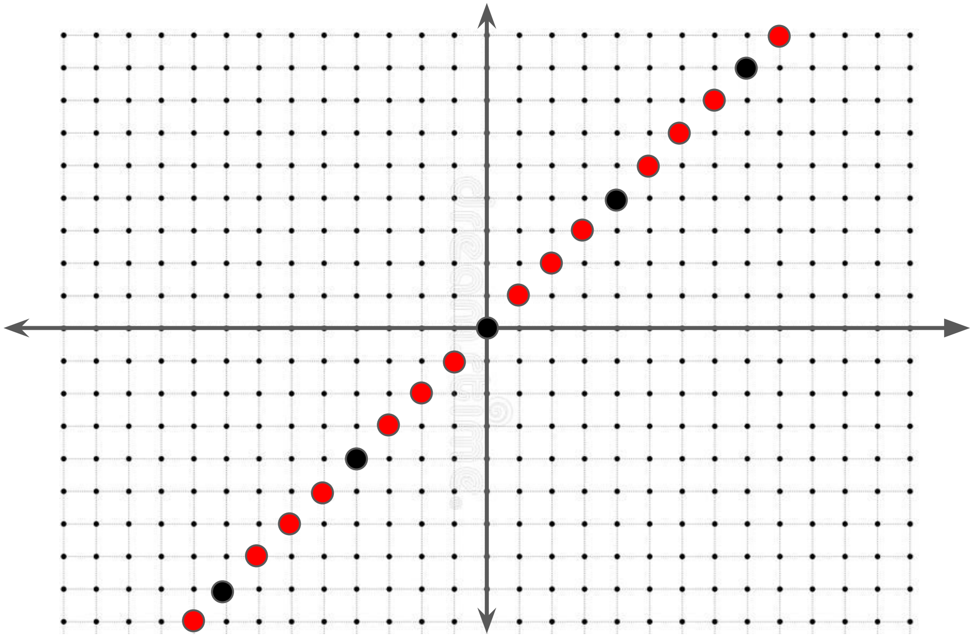

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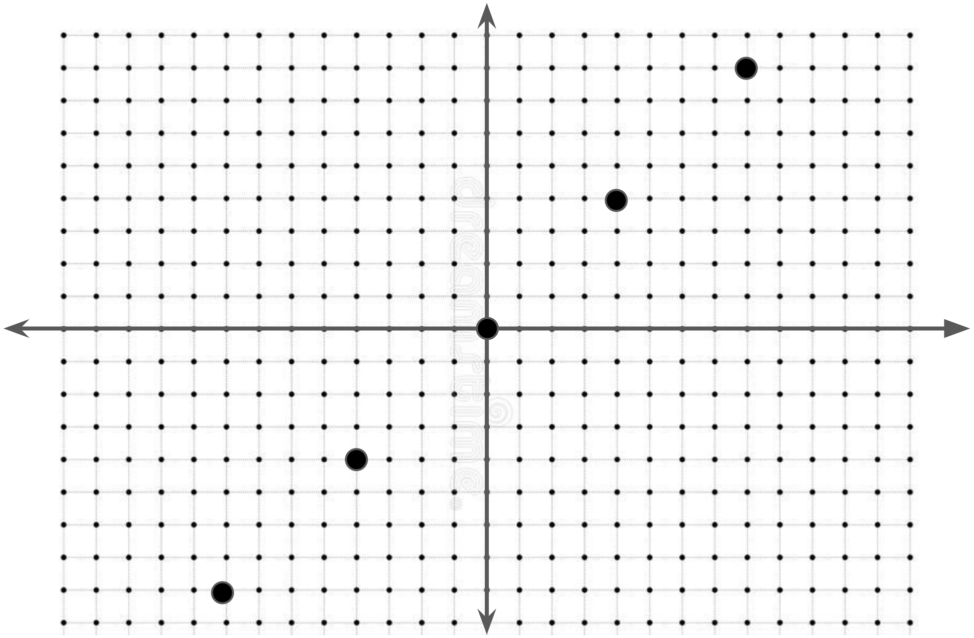

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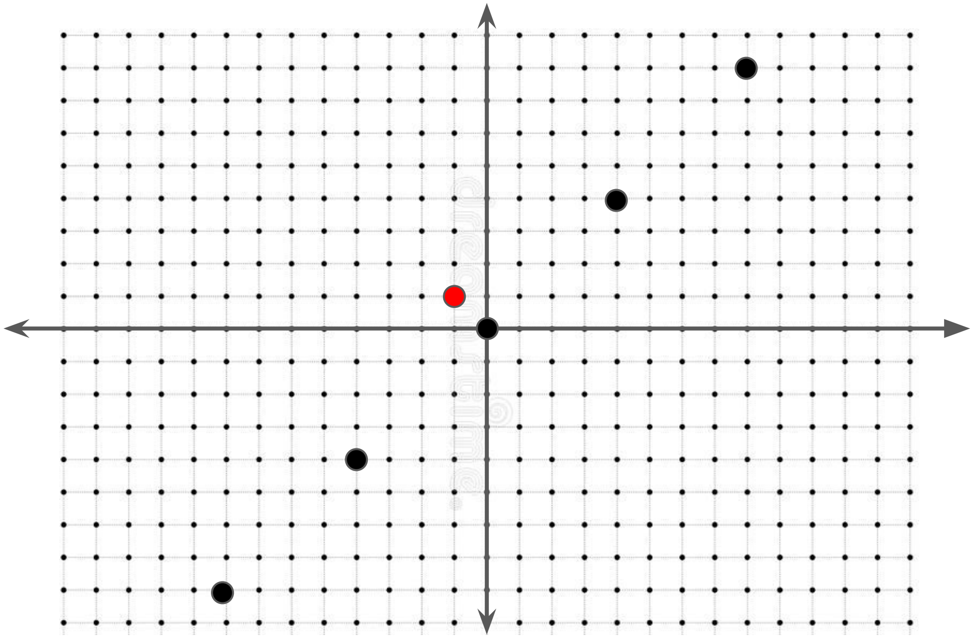

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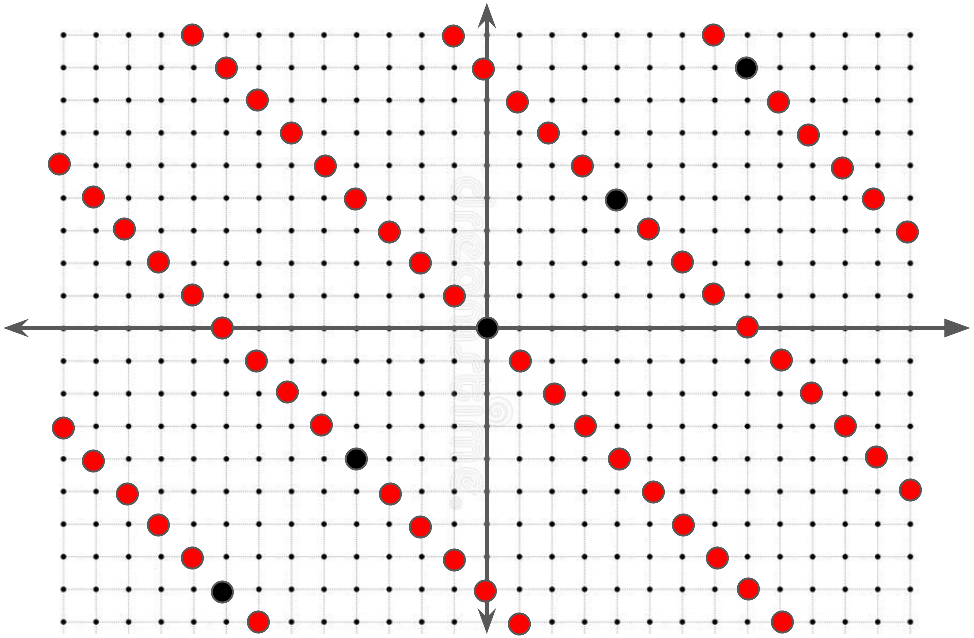

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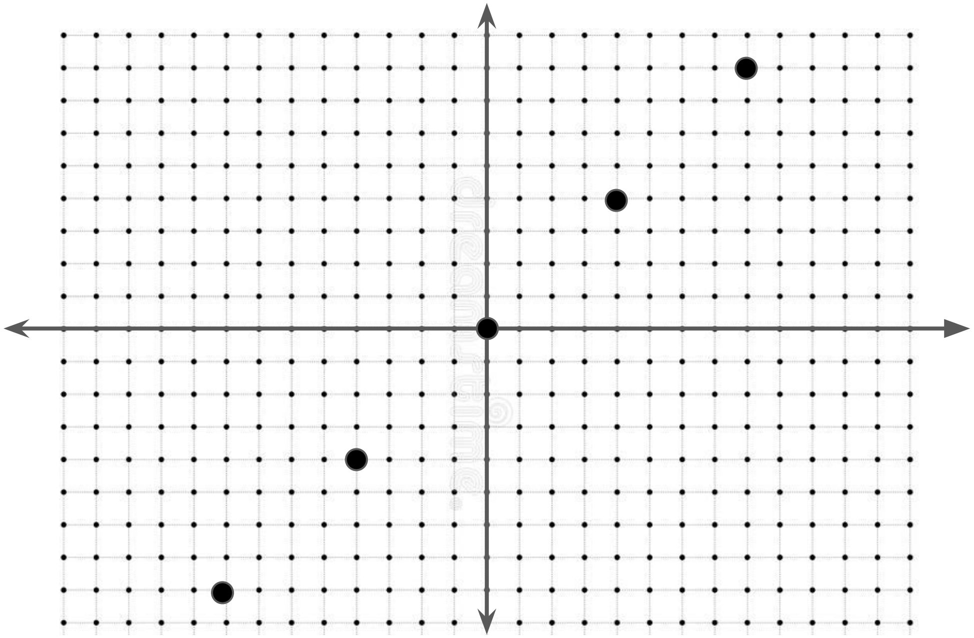

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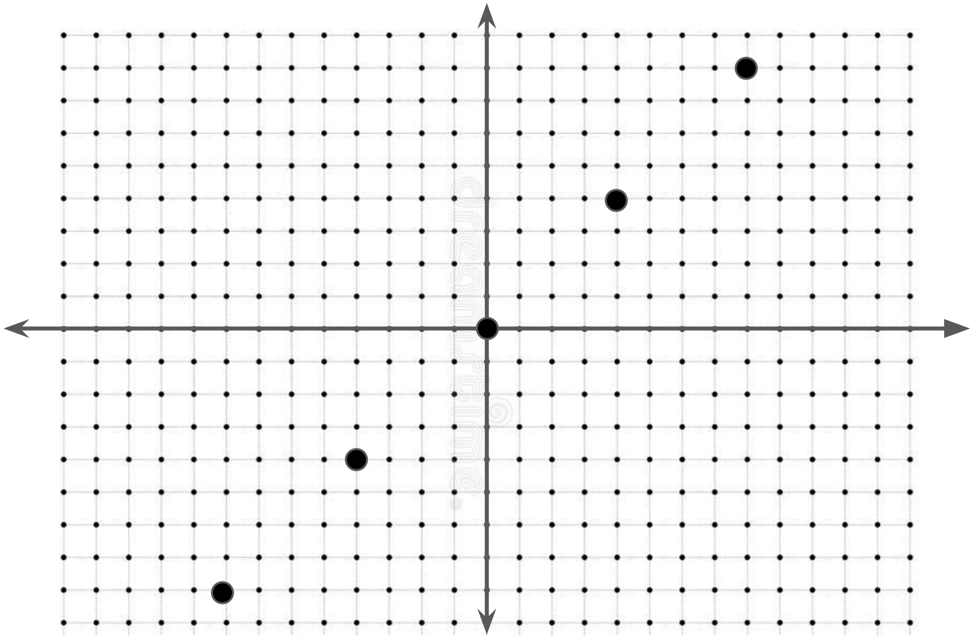

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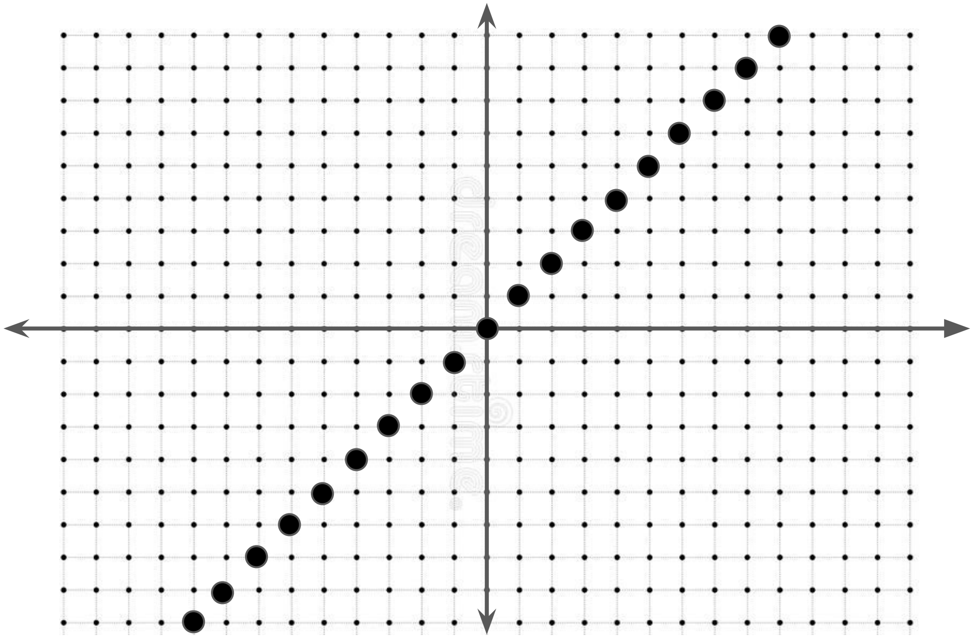

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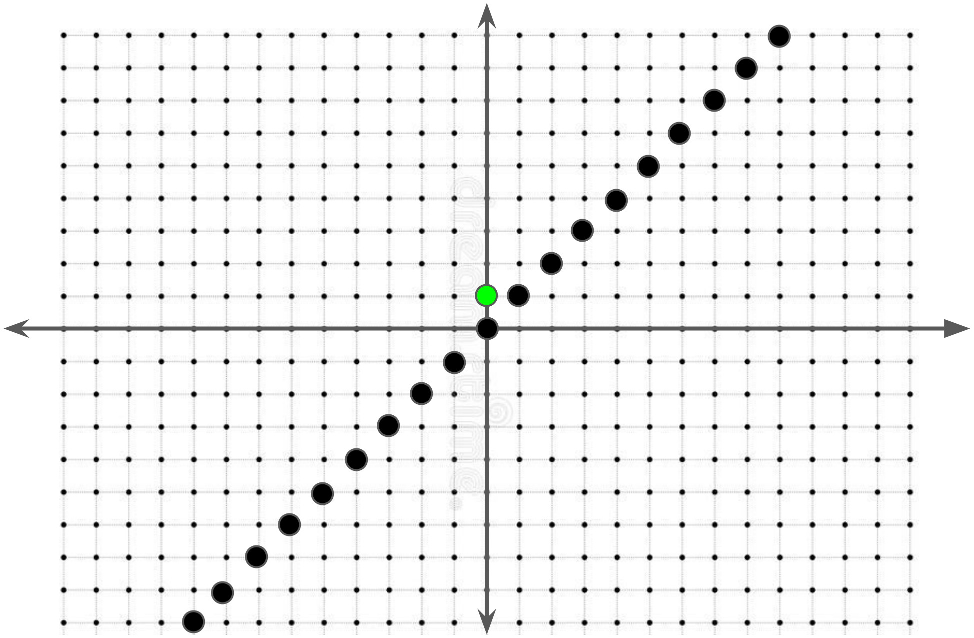

$$\boxed{\langle d+e \rangle \oplus \langle d \rangle \cong \mathbb{Z}^2}$$

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Something to look at...

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Divisor-enclosed enlargements of V_3, V_1 in $V_1 \otimes V_2$:

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Consider:

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Divisor-enclosed enlargements of $V_3, \; V_1$ in $V_1 \otimes V_2$:

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Consider:

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What kind of object does this form?

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Consider:

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What kind of object does this form?

Thanks!