Miniscule Representations of $SL(n, \mathbb{C})$ and Plane Partition Enumeration Problems

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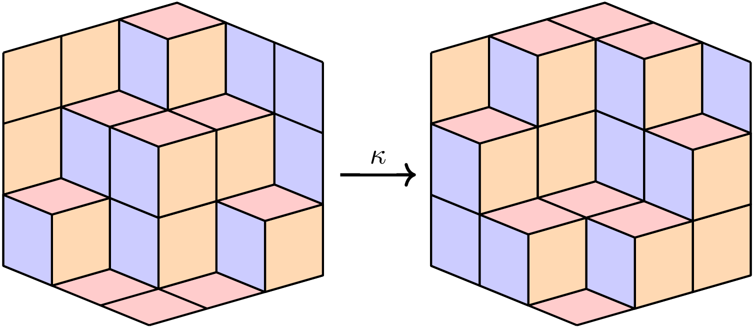
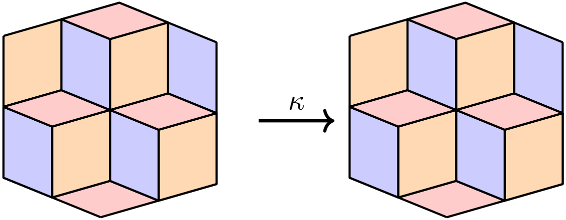
Interesting Enumeration Identities

Complementation:



Interesting Enumeration Identities

Complementation invariant partition:



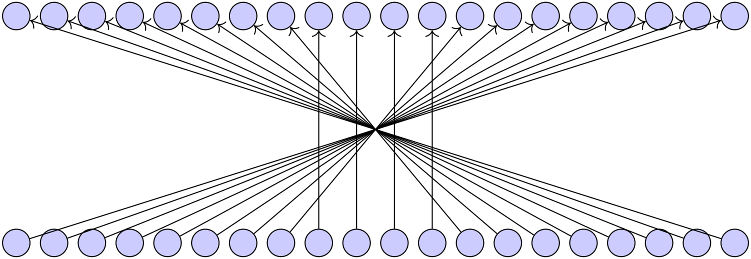
Interesting Enumeration Identities

$SCPP(2a, 2b, 2c)$ = number of self-complementary plane partitions in a $2a \times 2b \times 2c$ box

$PP(a, b, c)$ = number of plane partitions in an $a \times b \times c$ box

- $SCPP(2a, 2b, 2c) = PP(a, b, c)^2$
- $SCPP(2a, 2b, 2c) = PP_{-1}(2a, 2b, 2c)$

A Different Way to Think of Things...



Enter Representation Theory

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Method: find a convenient Lie group representation

Enter Representation Theory

A **Lie group** is a group that is also a smooth surface.

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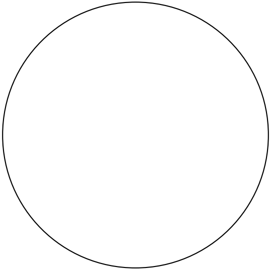
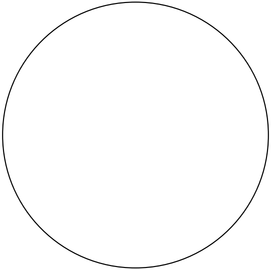


Enter Representation Theory

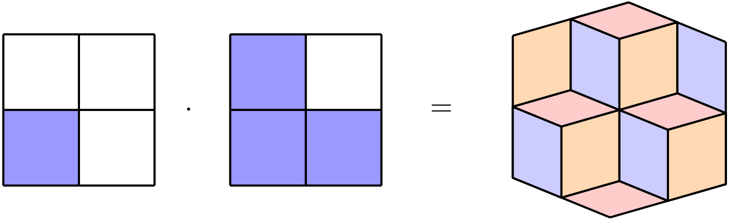
A **Lie group** is a group that is also a smooth surface.



$SL(n, \mathbb{C})$ is an example of a Lie group, and it happens to be exactly the group we want.

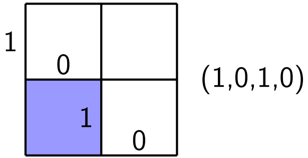
Constructing the Right Vector Space

Every plane partition can be viewed as a chain of c rectangle partitions.



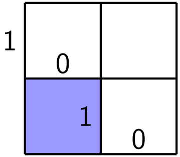
Constructing the Right Vector Space

Every rectangle partition can be viewed as a binary sequence of length $a + b$.



Constructing the Right Vector Space

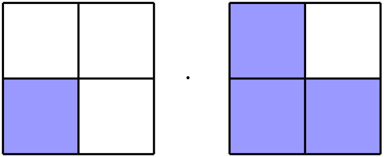
Let V be a vector space with $a + b$ basis elements.
Take the space of antisymmetric polynomials of degree a over V, $\Lambda^a V$.
By definition, $x_i \wedge x_j = -x_j \wedge x_i$.



$$x_1^1 \wedge x_2^0 \wedge x_3^1 \wedge x_4^0 = x_1 \wedge x_3$$

Constructing the Right Vector Space

Take the space of symmetric polynomials of degree c over $\Lambda^a V$, $S^c(\Lambda^a V)$.



$= (x_1 \wedge x_3)(x_2 \wedge x_4)$

Constructing the Right Vector Space

All of these spaces are representations of $SL(a+b, \mathbb{C})$.

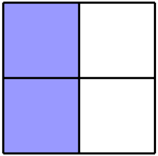
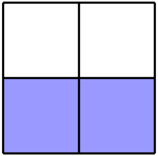
$$SL(a+b)$$

$$GL(V) \qquad GL(\Lambda^a V) \qquad GL(S^c(\Lambda^a V))$$

$$\{x_1, \dots, x_{a+b}\} \qquad \{\wedge_{i=1}^{a} x_{\alpha_i}\} \qquad \{\Pi_{j=1}^{c} (\wedge_{i=1}^{a} x_{\alpha_i})\}$$

Constructing the Right Vector Space

But...

 $\cdot$  $=$ not a plane partition

Constructing the Right Vector Space

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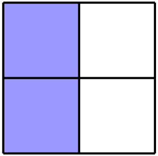
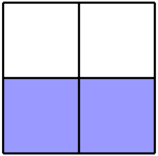
 $\cdot$  $=$ not a plane partition

New goal: Find a representation of $SL(a+b)$ inside $S^c(\Lambda^a V)$ with only plane partitions for basis elements

Irreducible Miniscule Lattices

Which combinations of rectangle partitions work?

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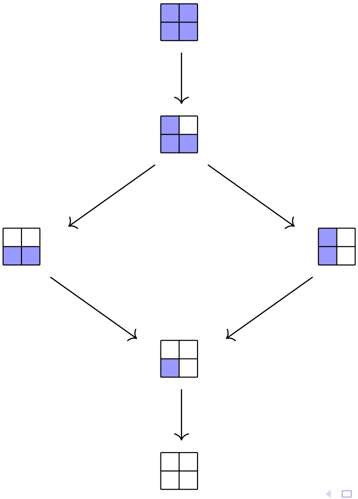

Irreducible Miniscule Lattices

Irreps are uniquely associated with lattices of basis vectors.

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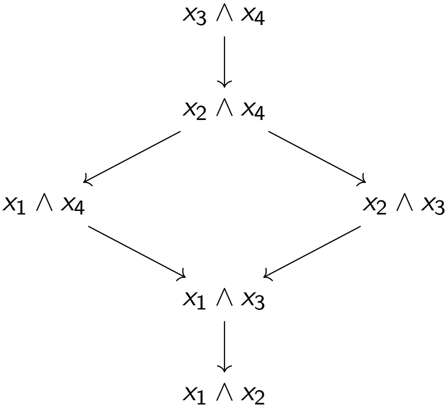
Lattice associated with $\Lambda^2 V$ as a representation of $SL(2+2)$:



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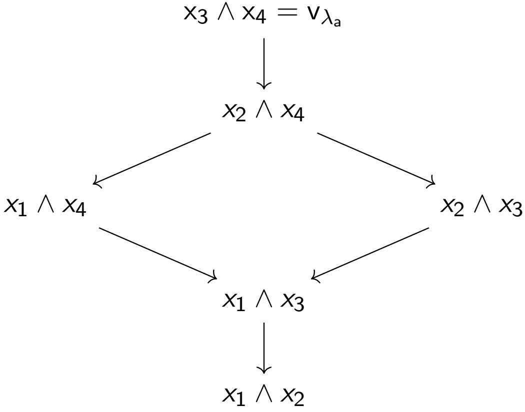
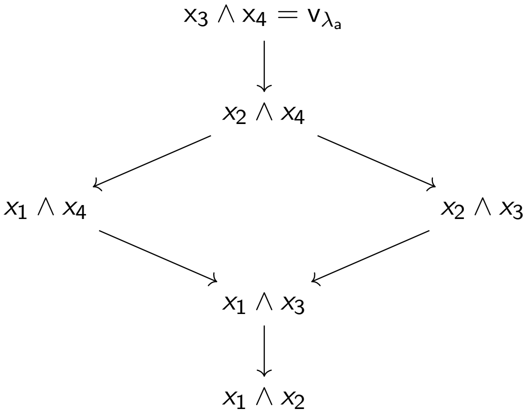


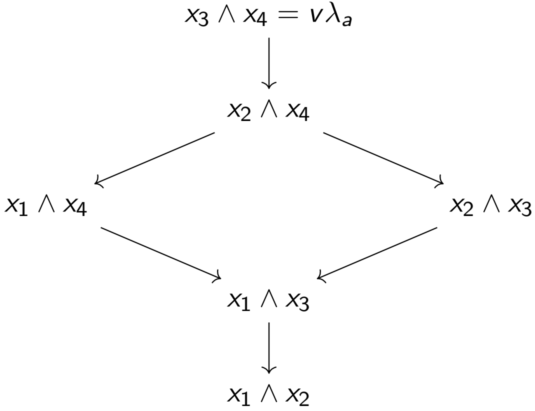
Irreducible Miniscule Lattices

Theorem: These partial orders are the same!
Lattice associated with $\Lambda^2 V$ as a representation of $SL(2+2)$:



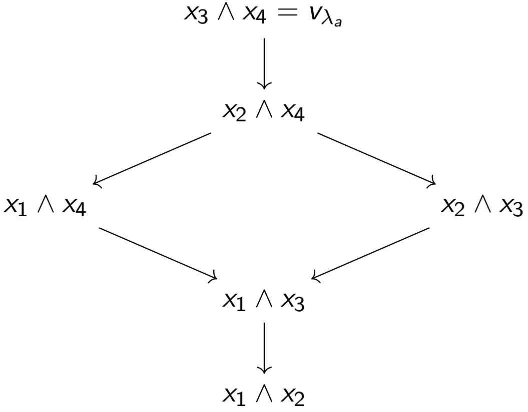
Irreducible Miniscule Lattices

Special case: Irrep $V(c\lambda_a)$ with highest weight vector $c\lambda_a$



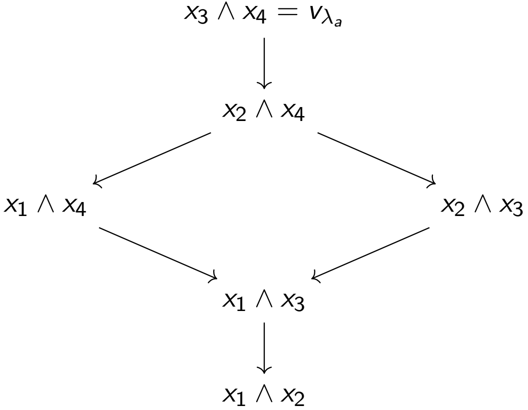
Irreducible Miniscule Lattices

Theorem: Basis elements of $V(c\lambda_a)$ are c-multichains in the lattice for $V(\lambda_a)$.



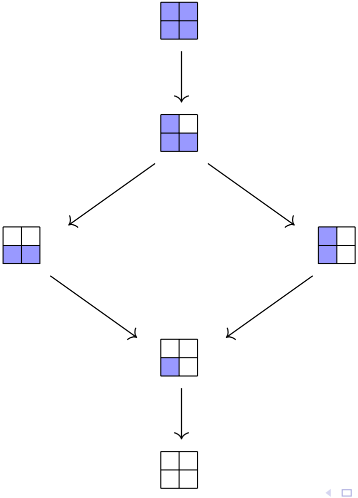
Irreducible Miniscule Lattices

Basis elements: $\{(x_3 \wedge x_4)(x_3 \wedge x_4), (x_1 \wedge x_3)(x_2 \wedge x_4), \ldots\}$
Not a basis element: $(x_1 \wedge x_4)(x_2 \wedge x_3)$



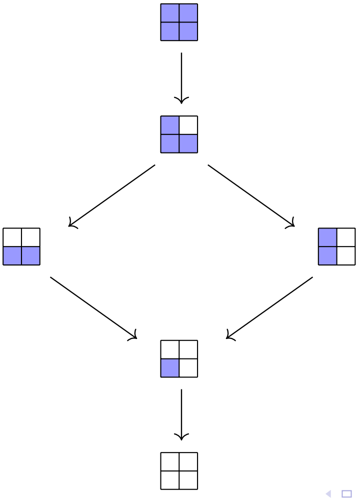
Relationship to Plane Partitions

These c-multichains exactly correspond to plane partitions!



Relationship to Plane Partitions

So, $V(c\lambda_a) \subseteq S^c(\Lambda^a V)$ is the representation we want.



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Now that we have this representation, how can we use it?

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- Let $A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

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- So, $tr(A_1)$ in its action on $V(2\lambda_2)$ is the sum of all basis elements.
- Can calculate this sum using the Weyl dimension formula

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- $A_{-1} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & -1 \end{bmatrix} \in SL(2a+2b)$ has trace $PP_{-1}(2a, 2b, 2c)$

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- The trace of $B = \begin{bmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ . & \dots & . & . \\ 1 & \dots & 0 & 0 \end{bmatrix}$ in $V(2c\lambda_{2a})$ is $SCPP(2a, 2b, 2c)$.

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A_{-1} is conjugate to B, so $PP_{-1}(2a, 2b, 2c) = SCPP(2a, 2b, 2c)$, as desired.

Thank You!