Sum and Product Games

Longest Chains and Other Counts

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July 29 2022

UC Davis

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Set Up

Notation

$$
\begin{aligned}
G(n) &: \text{ Graph with max value } n \\
V_p(n) &: \text{ Product nodes in } G(n) \\
V_s(n) &: \text{ Sum nodes in } G(n) \\
d(V(n)) &: \text{ Degree of node in } G(n) \\
V_{p,d=3}(n) &: \text{ Condition on node (degree is equal to 3).}
\end{aligned}
$$

Basic Counts

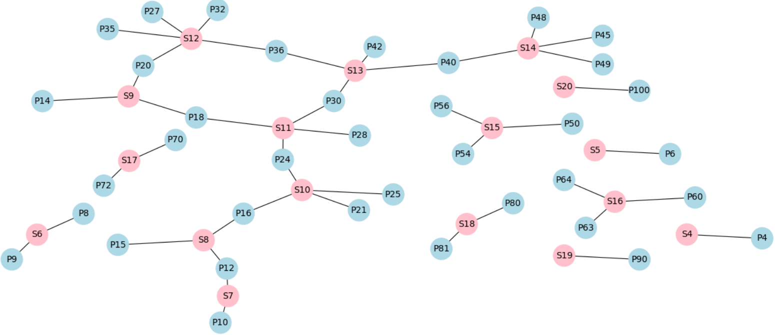


Figure 1: n = 10

Basic Counts

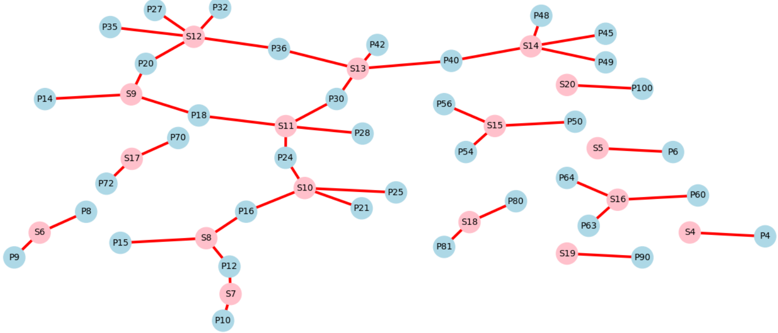


Figure 2: Number of edges = $\frac{n(n-1)}{2}$

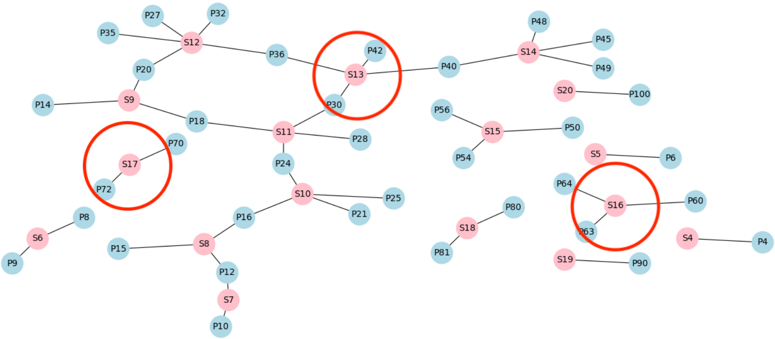


Figure 3: Degree of sum node = $\min(n - \lceil \frac{V_S}{2} \rceil + 1, \lfloor \frac{V_S}{2} \rfloor - 1)$

Basic Counts

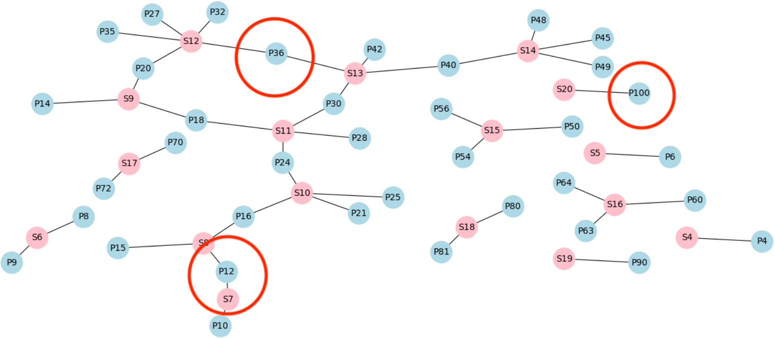


Figure 4: Degree of product node = $\left\lceil \frac{\prod_{i=1}^{m}(a_i+1)}{2} \right\rceil$

Basic Counts

$$\text{Number of edges} \ : \binom{n-1+2-1}{2} = \frac{n(n-1)}{2}$$

$$\text{Degree of sum node} \ : \min(n - \left\lceil \frac{V_s}{2} \right\rceil + 1, \left\lfloor \frac{V_s}{2} \right\rfloor - 1)$$

$$\text{Final degree of product node} \ : V_p = p_1^{a_1} p_2^{a_2} \dots p_m^{a_m}$$

$$d(V_p) = \left\lceil \frac{\prod_{i=1}^{m}(a_i+1)}{2} \right\rceil$$

Longest Chains

Big Question

What is the longest possible chain length in $G(n)$ for a given n
What is the maximum chain length in $G(n)$ for ANY n?

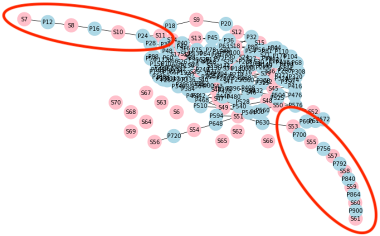


Figure 5: n = 36

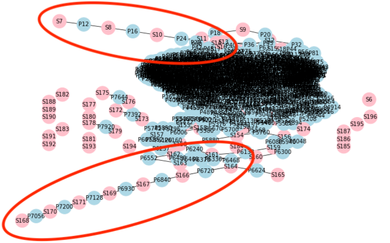


Figure 6: n = 99

Data

n	max
10	7
11	7
12	7
13	7
14	6
15	6
16	6
17	6
18	6
19	6
20	6
21	7
22	6

n	max
23	6
24	6
25	9
26	6
27	6
28	6
29	6
30	6
31	6
32	8
33	6
34	6
35	6

n	max
36	13
37	13
38	11
39	6
40	6
41	6
42	6
43	6
44	6
45	6
46	6
47	6
48	6

n	max
49	7
50	6
51	6
52	6
53	6
54	6
55	6
56	6
57	7
58	7
59	7
60	6
61	6

Data

n	max
10	7
11	7
12	7
13	7
14	6
15	6
16	6
17	6
18	6
19	6
20	6
21	7
22	6

n	max
23	6
24	6
25	9
26	6
27	6
28	6
29	6
30	6
31	6
32	8
33	6
34	6
35	6

n	max
36	13
37	13
38	11
39	6
40	6
41	6
42	6
43	6
44	6
45	6
46	6
47	6
48	6

n	max
49	7
50	6
51	6
52	6
53	6
54	6
55	6
56	6
57	7
58	7
59	7
60	6
61	6

Data

n	max
10	7
11	7
12	7
13	7
14	6
15	6
16	6
17	6
18	6
19	6
20	6
21	7
22	6

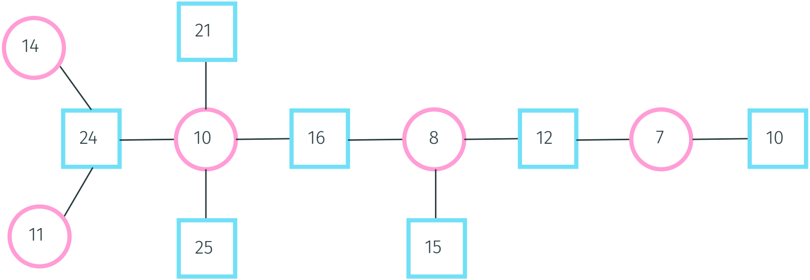
n	max
23	6
24	6
25	9
26	6
27	6
28	6
29	6
30	6
31	6
32	8
33	6
34	6
35	6

n	max
36	13
37	13
38	11
39	6
40	6
41	6
42	6
43	6
44	6
45	6
46	6
47	6
48	6

n	max
49	7
50	6
51	6
52	6
53	6
54	6
55	6
56	6
57	7
58	7
59	7
60	6
61	6

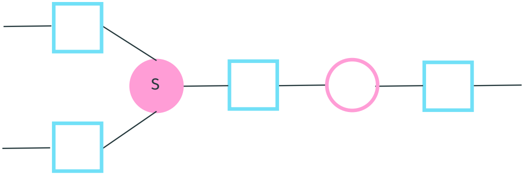
Why is 6 Frequent?

This is the only permanent chain at small product nodes, length is 6

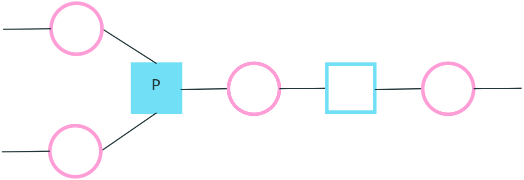


Why are Odds Frequent?

Other than the 6 chain, all but 3 graphs (for $10 \leq n \leq 1000$) have odd lengths.



A chain is odd if it starts with a sum



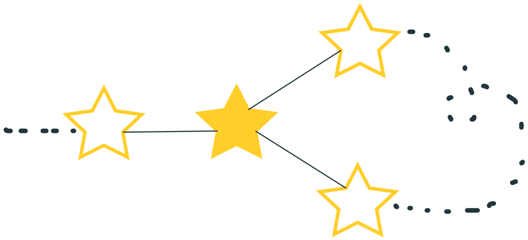
A chain is even if it starts with a product

Conditions to Start a Chain

Goal: find the probability that start of a chain is a product versus a sum.

Conditions to start a chain:

- Node is on the *edge* of the graph ($V_s(n) \geq 2n - k\sqrt{n}$)
- Degree at least 3
- At least 3 neighbors of the node must have degree at least 2



Need to Count

In order to calculate the probabilities we need the following:

- Total number of sum nodes
- Distribution of degrees in sum nodes
- Total number of product nodes
- Distribution of degrees in product nodes

Counting

Sum Nodes

Every sum from 4 to $2n$ is achievable as a sum in $G(n)$

Thus:

$$|V_s(n)| = 2n - 3$$

Possible degrees of sum nodes: $1, 2, \ldots \left\lfloor \frac{n}{2} \right\rfloor$

Frequency of each degree: For all but highest degree, 4 nodes of each degree occur.

Total Number of Product Nodes

Product nodes range from 4 to n^2

Total Number of Product Nodes

Unachievable product nodes (n=10):

Primes less then n

Total Number of Product Nodes

Unachievable product nodes (n=10):

Primes less then n

Primes between n and n^2

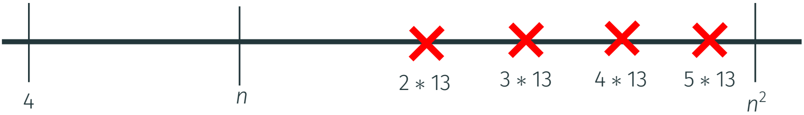
Total Number of Product Nodes

Unachievable product nodes (n=10):

Primes less then n

Primes between n and n^2

All multiples of primes between n and n^2



Total Number of Product Nodes

Product nodes range from 4 to n^2

Unachievable product nodes (n=10):

- Primes less then n (eg. $1 * 5$)
- Primes between n and n^2 (eg. $1 * 17$)
- All multiples of primes between n and n^2
- Three primes such that each partitioning results in an element larger than n

Total Number of Product Nodes

Product nodes range from 4 to n^2

Unachievable product nodes (n=10):

- Primes less then n (eg. $1 * 5$)
- Primes between n and n^2 (eg. $1 * 17$)
- All multiples of primes between n and n^2
- Three primes such that each partitioning results in an element larger than n
- Four primes????

Erdös-Kac Theorem

If $\omega(n)$ is number of distinct prime factors of n, then the probability distribution is:

$$\frac{\omega(n) - \log\log(n)}{\sqrt{\log\log n}}$$

n will have an average of 4 distinct primes when
$n = 1,000,000,000,000,000,000,000,000$

Total Number of Product Nodes

We wish to estimate the number of unachievable nodes:

1. Primes less than n

We know the the density of primes less than n is $\frac{1}{\ln n}$

We know the number of primes less than n can be approximated by:

$$\pi(n) = \frac{n}{\ln n}$$

Total Number of Product Nodes

We wish to estimate the number of unachievable nodes:

2 and 3. Primes and multiples between n and n^2

To find all primes and their multiples between n and n^2, we solve:

$$\int_{n}^{n^2} \frac{1}{\ln p} * \frac{n^2}{p} dp \;=\; n^2(\ln 2)$$

Total Number of Product Nodes

We wish to estimate the number of unachievable nodes:

4. Product of 3 primes between 2 and n

We want 3 primes x, y, z such that:

$$xyz \leq n^2 \qquad xy > n$$

$$xz > n \qquad yz > n$$

We take the natural log of everything:

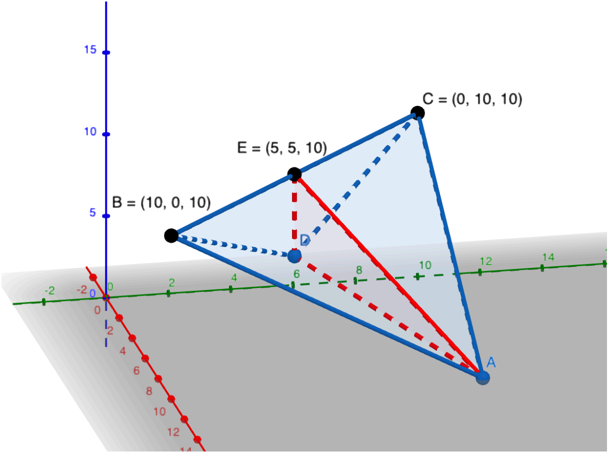
$$\tilde{x} + \tilde{y} + \tilde{z} \leq 2\tilde{n} \qquad \tilde{x} + \tilde{y} > \tilde{n}$$

$$\tilde{x} + \tilde{z} > \tilde{n} \qquad \tilde{y} + \tilde{z} > \tilde{n}$$

We wish to estimate the number of unachievable nodes:

4. Product of 3 primes between 2 and n

Solution to system of linear equations:



Total Number of Product Nodes

We wish to estimate the number of unachievable nodes:

4. Product of 3 primes between 2 and n

To count the number of triples x, y, z that satisfy our equations, we solve the triple integral:

$$\int_{\tilde{n}/2}^{\tilde{n}-1} \int_{\tilde{n}-\tilde{x}}^{\tilde{x}} \int_{\tilde{n}-\tilde{y}}^{2\tilde{n}-\tilde{x}-\tilde{y}} \frac{e^{\tilde{x}+\tilde{y}+\tilde{z}}}{\tilde{x}\tilde{y}\tilde{z}} \, d\tilde{x}\, d\tilde{y}\, d\tilde{z}$$

Total Number of Product Nodes

$$n^2 - 1 \quad - \quad \frac{n}{\ln n} \quad - \quad n^2(\ln 2) \quad - \quad \int_{\tilde{n}/2}^{\tilde{n}-1} \int_{\tilde{n}-\tilde{x}}^{\tilde{x}} \int_{\tilde{n}-\tilde{y}}^{2\tilde{n}-\tilde{x}-\tilde{y}} \frac{e^{\tilde{x}+\tilde{y}+\tilde{z}}}{\tilde{x}\tilde{y}\tilde{z}} d\tilde{x} d\tilde{y} d\tilde{z}$$

Total Number of Product Nodes

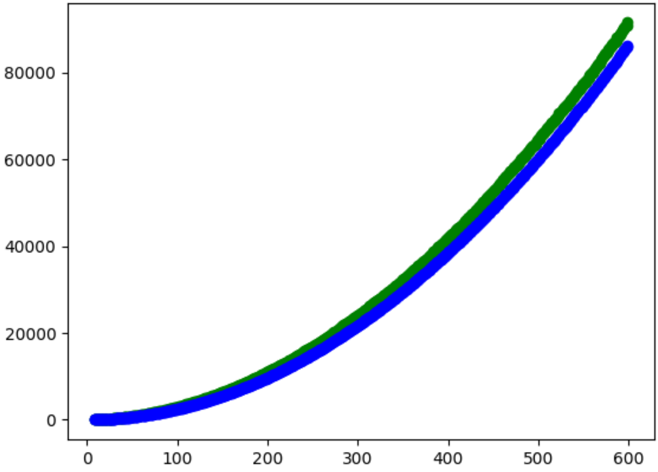


Figure 8: Real vs Estimate number of product nodes for 10 < n < 600

Erdös Conjecture

Fix $\delta \leq 1$. Then for a finite $A \subset \mathbb{Z}$, one has

$$|A + A| + |AA| \gtrsim |A|^{1+\delta}$$

In our case: $A = \{2, 3, \ldots n\}$

Distribution of Product Degrees

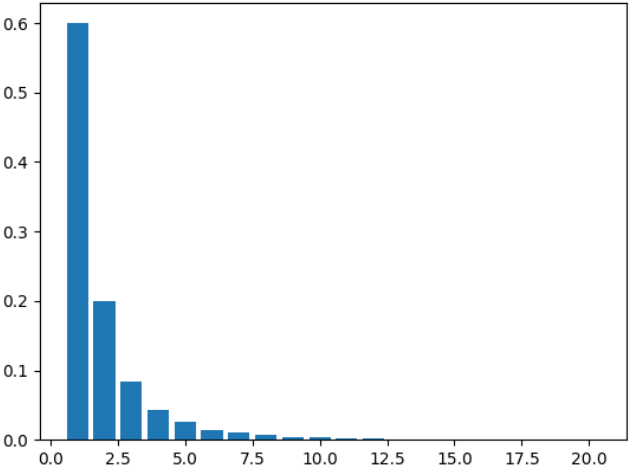


Figure 9: Normalized distribution of degrees for n = 500

It’s Exponential!

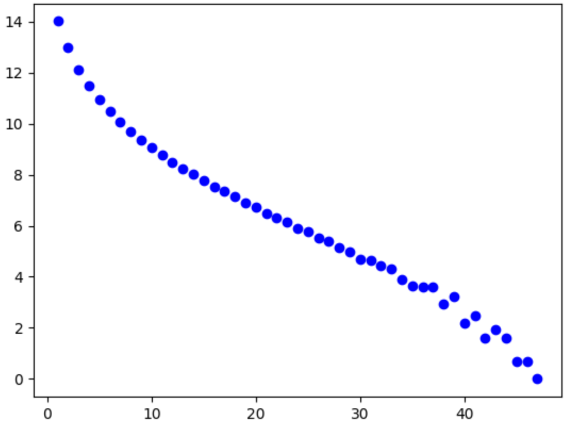


Figure 10: n = 3000, x, log y

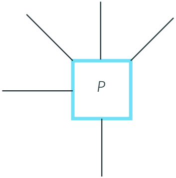
Markov Chain

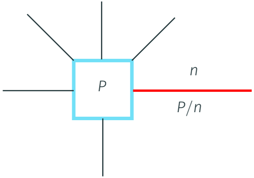
Begin with base case:
Distribution of degrees at n = 50

$$\begin{bmatrix} 518 \\ 162 \\ 104 \end{bmatrix}$$

Want to find distribution at $n = 51$:

When n increases by 1:





Markov Chain

$n = 51$

α: Number of new nodes in $G(n)$
β_1: Prob a node of degree 1 in $G(n-1)$ become degree 2 in $G(n)$
β_2: Prob a node of degree 2 in $G(n-1)$ become degree 3 in $G(n)$

$$\begin{bmatrix} 1-\beta_1 & \beta_1 & 0 \\ 0 & 1-\beta_2 & \beta_2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 518 \\ 162 \\ 104 \end{bmatrix} + \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}$$

Markov Chain

$$\alpha : |V_p(n)| - |V_p(n-1)|$$

$$\beta_1 : \frac{n-1-\alpha}{|V_p(n-1)|} * \left(1 - \frac{\binom{\pi(n)+1}{2}}{|V_p(n-1)|}\right)$$

$$\beta_2 : \frac{n-1-\alpha}{|V_p(n-1)|}$$

Markov Chain - Steady State

Iterate:

$$M = \begin{bmatrix} 518 \\ 162 \\ 104 \end{bmatrix}$$

$$M = \begin{bmatrix} 1-\beta_1 & \beta_1 & 0 \\ 0 & 1-\beta_2 & \beta_2 \\ 0 & 0 & 1 \end{bmatrix} \cdot M + \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}$$

Steady normalized state:

$$\begin{bmatrix} |V_{p,d=1}|/|V_p| \\ |V_{p,d=2}|/|V_p| \\ |V_{p,d\geq 3}|/|V_p| \end{bmatrix} = \begin{bmatrix} 0.58 \\ 0.20 \\ 0.22 \end{bmatrix} \approx \begin{bmatrix} 0.573 \\ 0.203 \\ 0.224 \end{bmatrix}$$

Probability

Putting it All Together

Probability that a sum node is the beginning of a chain:

$\alpha =$ average degree of sum nodes on edge.
$p_1 =$ probability that product node on the edge has degree at least 2.

$$(1 - bin(\alpha, 0, p_1) - bin(\alpha, 1, p_1) - bin(\alpha, 2, p_1))$$

Probability that a product node is the beginning of a chain:

$p_2 =$ probability that product node on the edge has degree at least 3.

$$(1 - bin(\alpha, 0, p_2) - bin(\alpha, 0, p_1))$$

Plugging in Values

Need to address that we are looking at nodes **at the edge**.
Probabilities are different than in the whole graph.

Edge is defined by $2n - k\sqrt{n}$, the parameter k determines how far out we look.

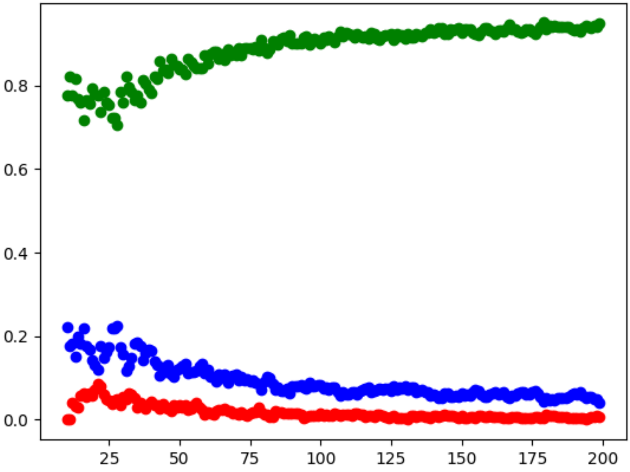
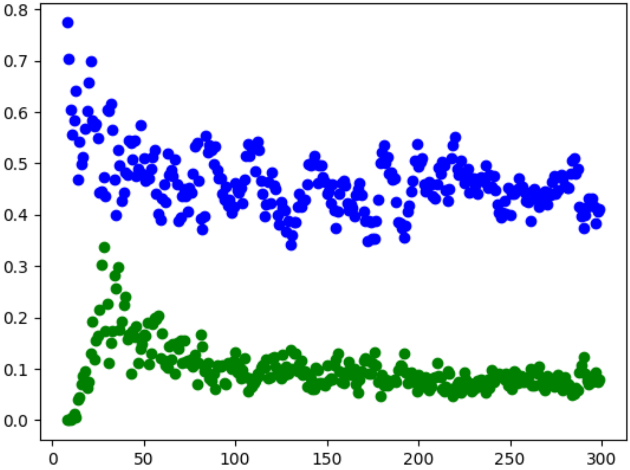


Figure 11: $k = 5 \quad 10 < n < 200$

Using this Data

Sum: $(1 - bin(\alpha, 0, p_1) - bin(\alpha, 1, p_1) - bin(\alpha, 2, p_1))$
Product: $(1 - bin(\alpha, 0, p_2) - bin(\alpha, 0, p_1))$



Thanks