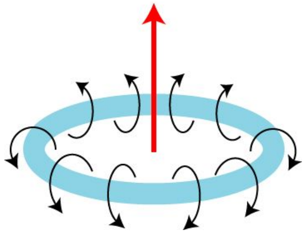
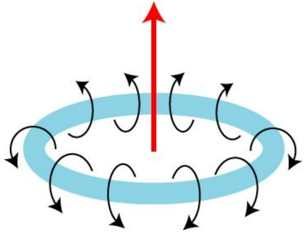
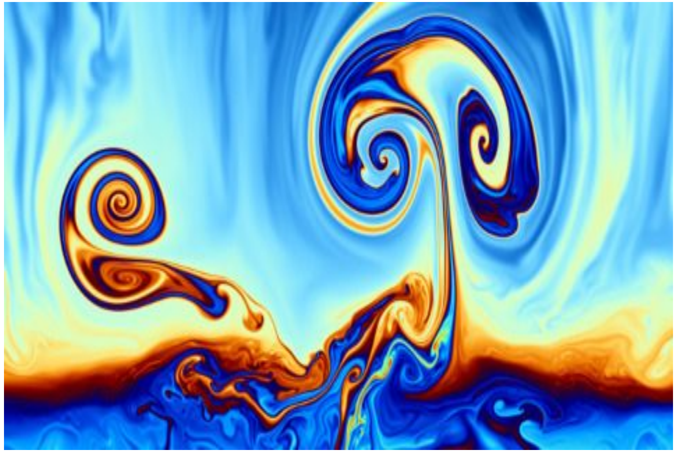
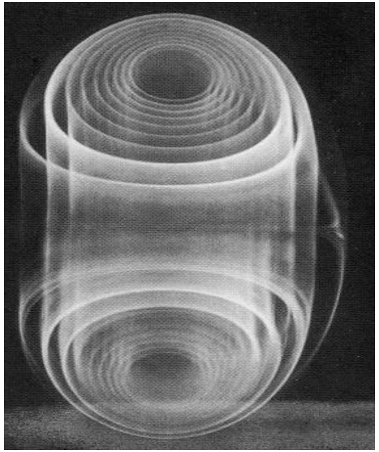
Incompressible Fluids, Non-traditional Coriolis terms, and Vortex Rings in a Rotating Frame of Reference





Shriya Fruitwala and Prezley Strait





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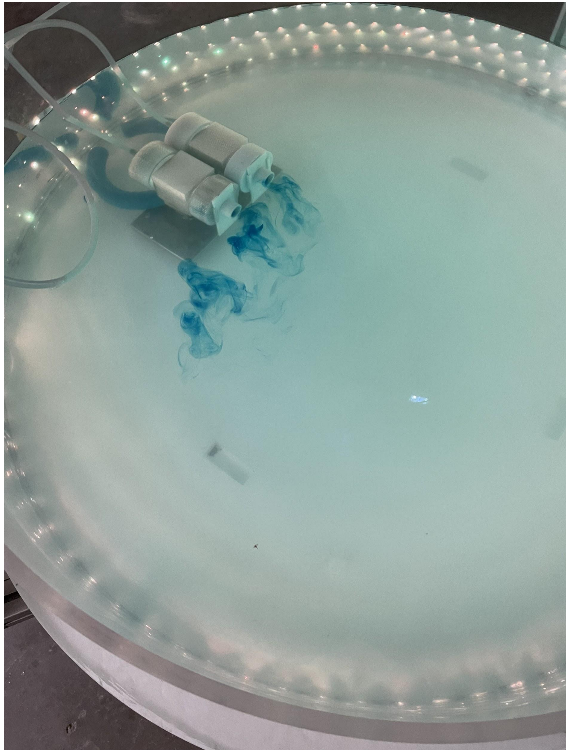
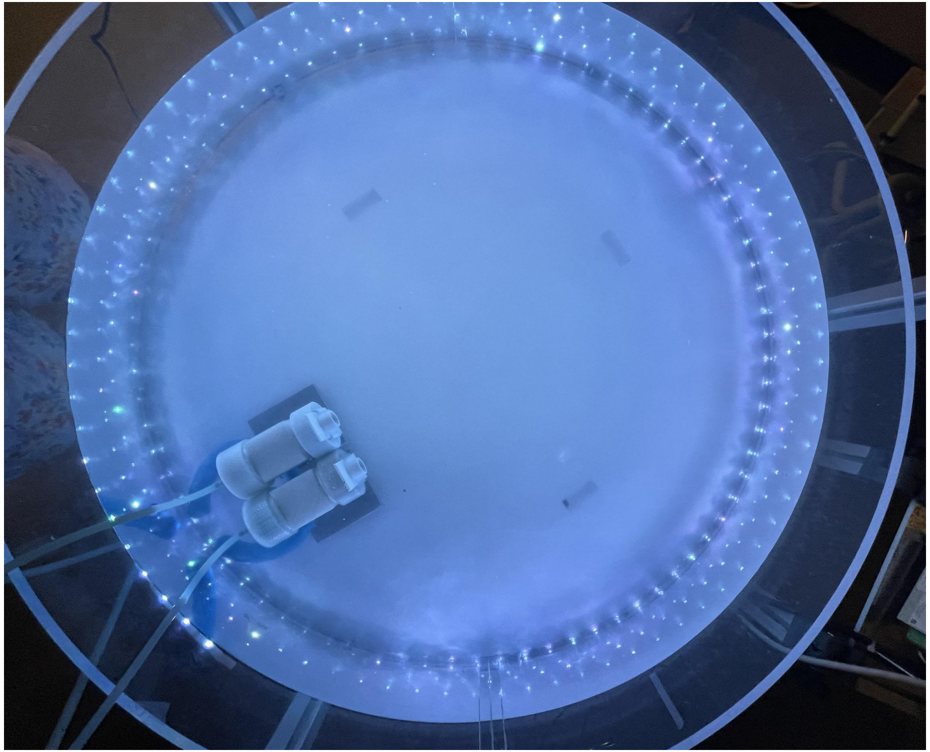
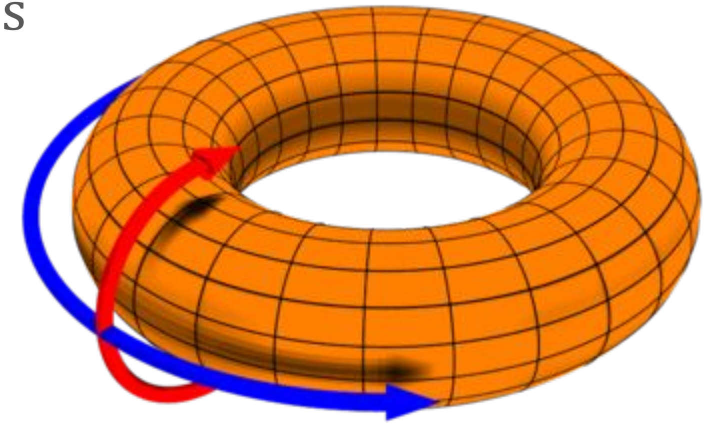
Introduction





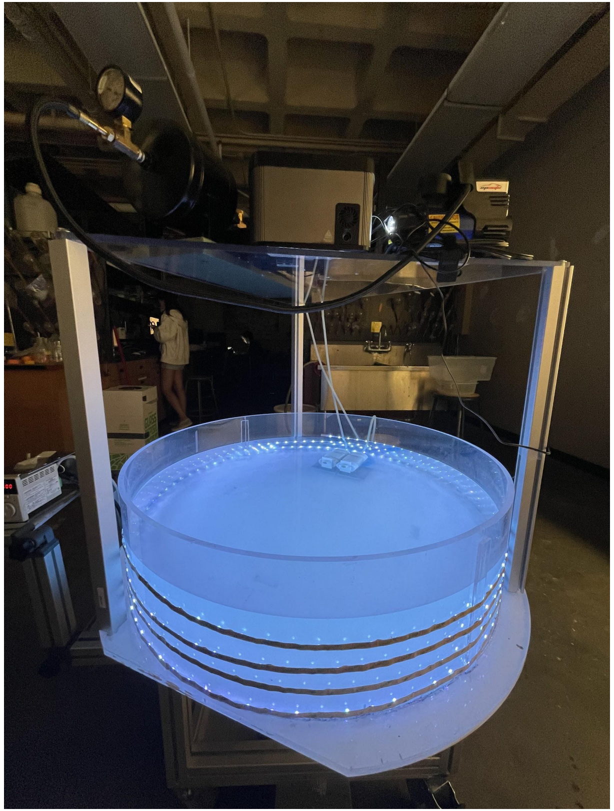
What is a vortex ring?

- Torus-shaped vortex in fluid
- Poloidal flow
- Fluid spins around imaginary axis that forms a closed loop



Purpose

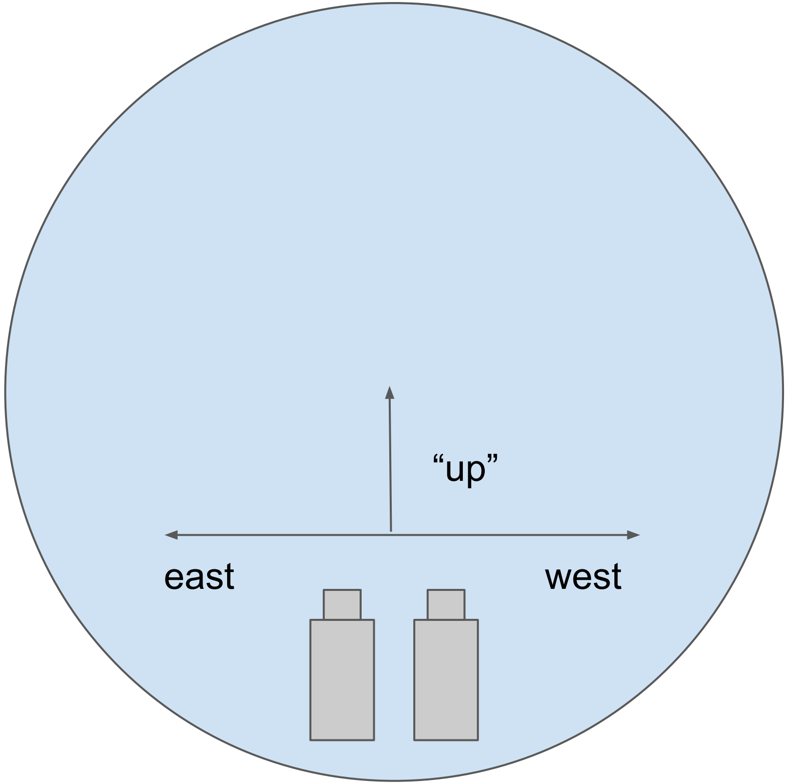
1. Understand the properties and dynamics of vortex rings
2. Detect a mean flow generated by the movement of the rings in the “west” direction



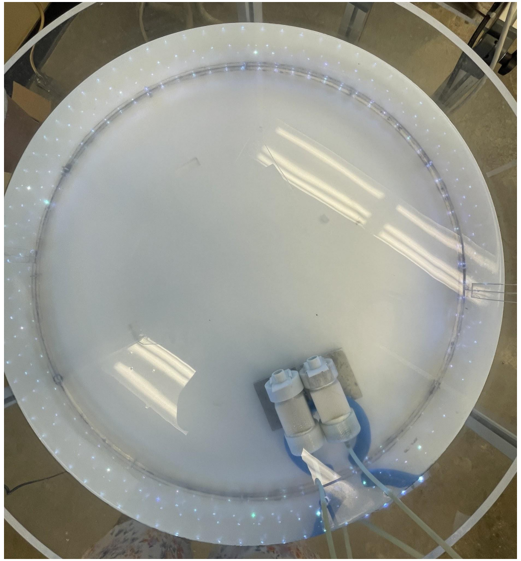
Bigger Picture

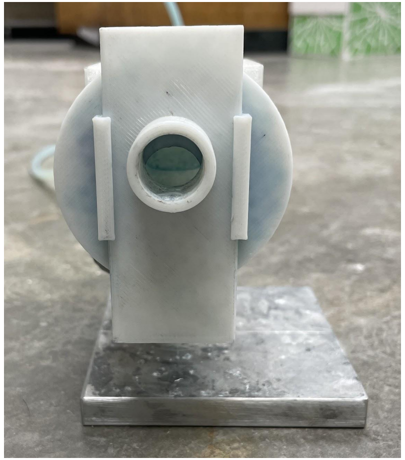
The experiment models cloud convection at the equator in the presence of the Coriolis force. The vortex rings represent air parcels that tilt to the west as they rise in the atmosphere due to the vertical component of the Coriolis force. By detecting a mean flow, we would confirm the theory that the vertical component of the Coriolis force plays a relevant role in equatorial atmospheric dynamics.

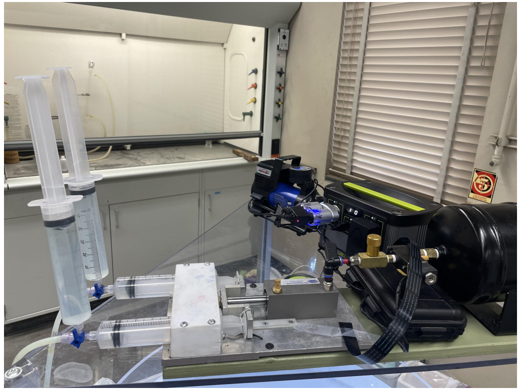
The Experiment - Orientation

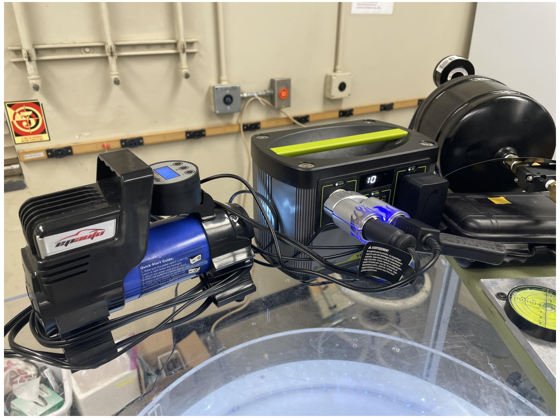


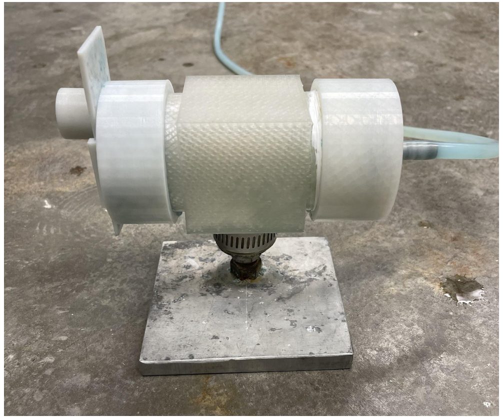
The Experiment - Overview









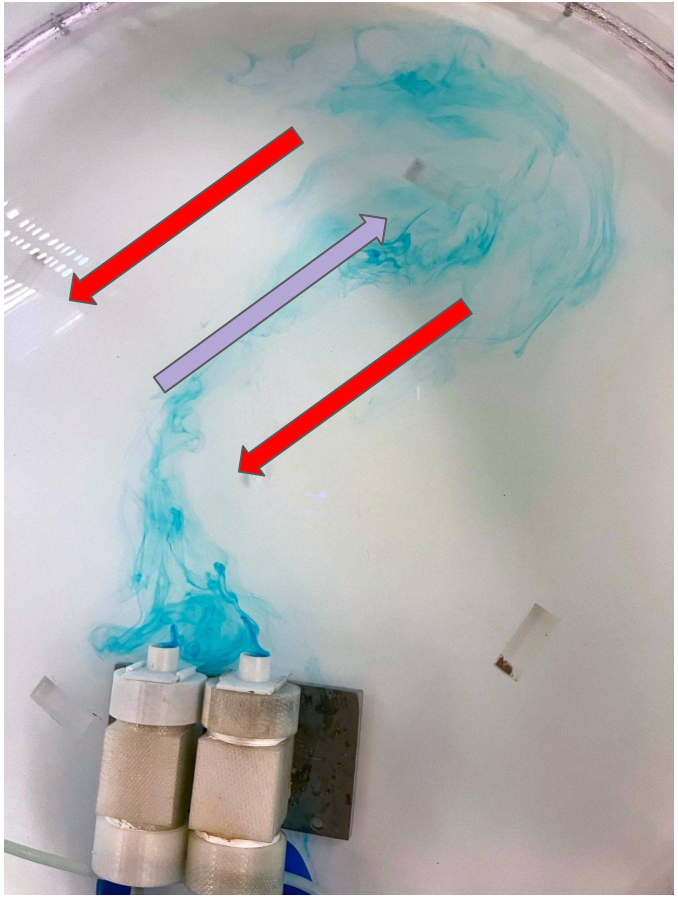


The Experiment - Overview

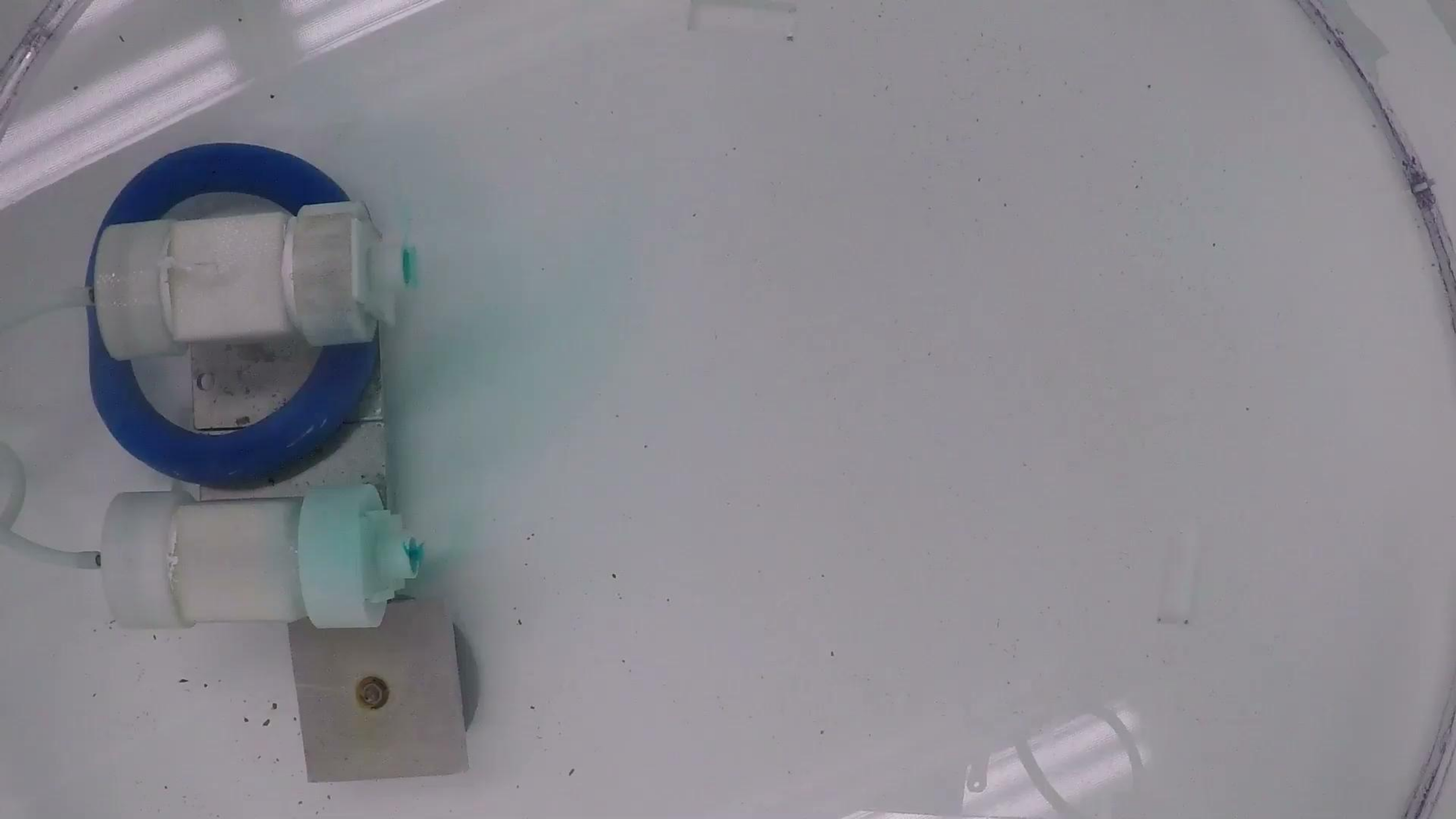
- The system includes two vortex cannons, which are launched using Python code
- Cannons filled with water dyed with methylene blue
- Parameters:
  - Duration, Wait, Repeat (in launching code)
  - Pressure, Valve (in physical setting)
- These parameters, along with our observations are recorded in an Excel sheet

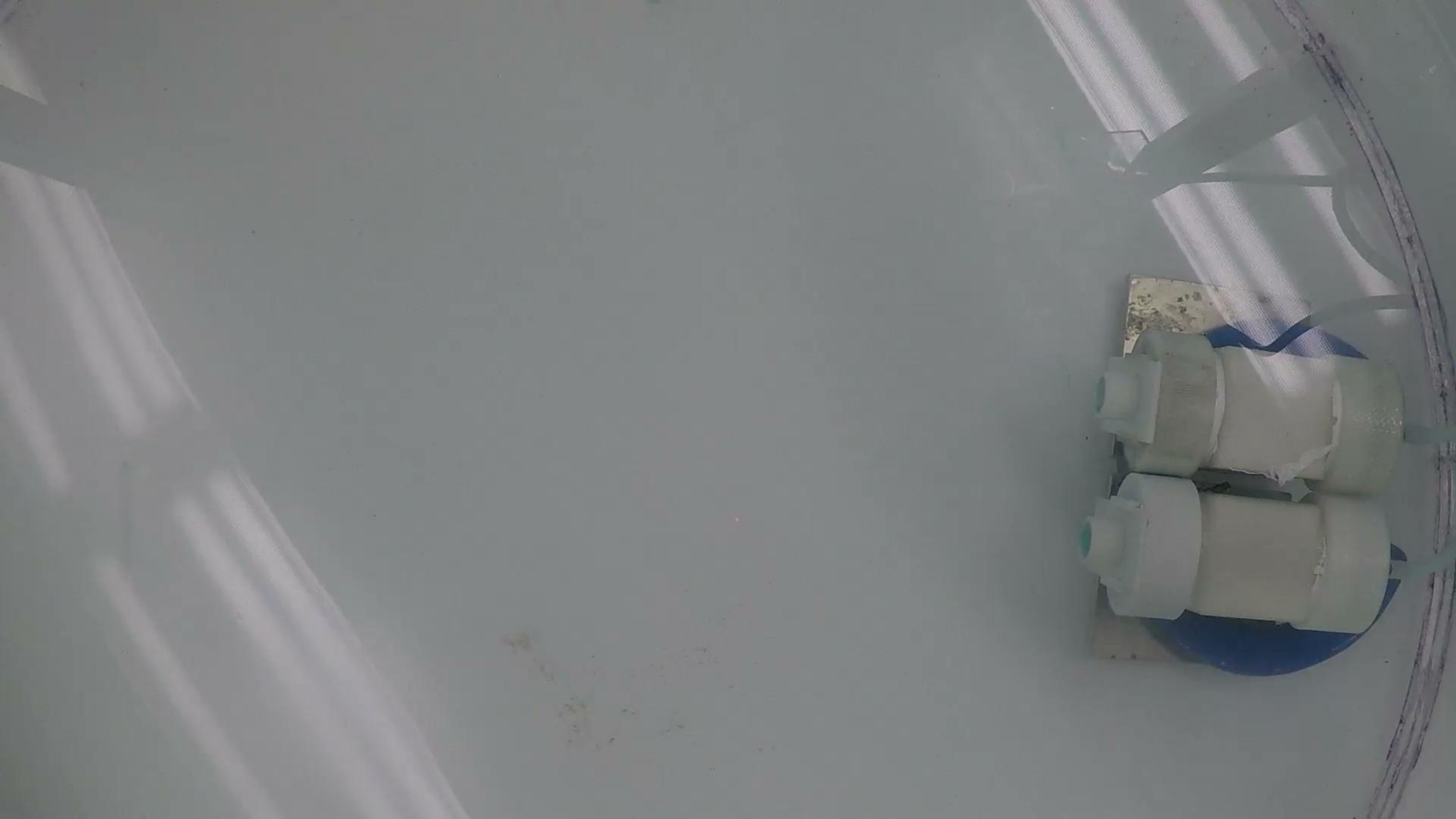
The Experiment - Overview

- Still tank experiments
- Rotating tank experiments
- Detecting mean flow (rheoscopic fluid)

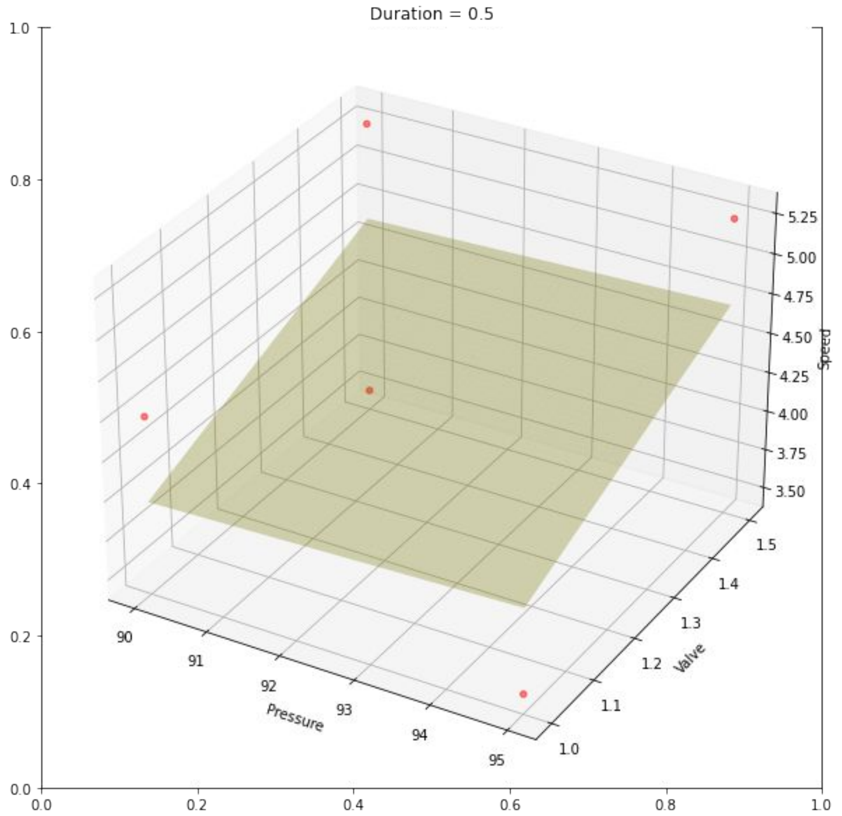


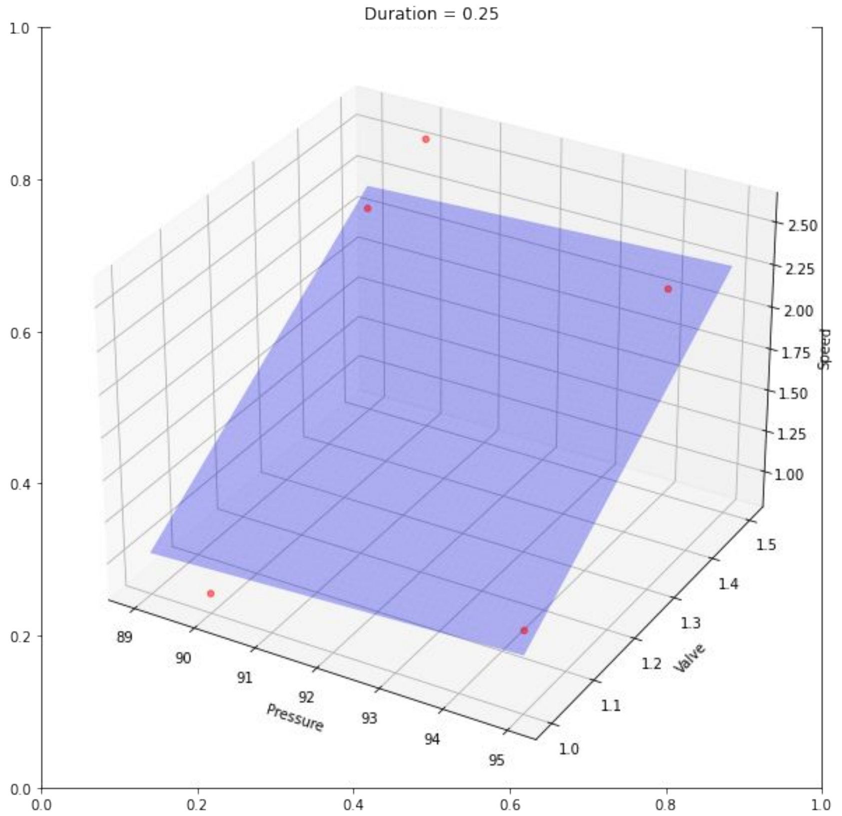
Videos





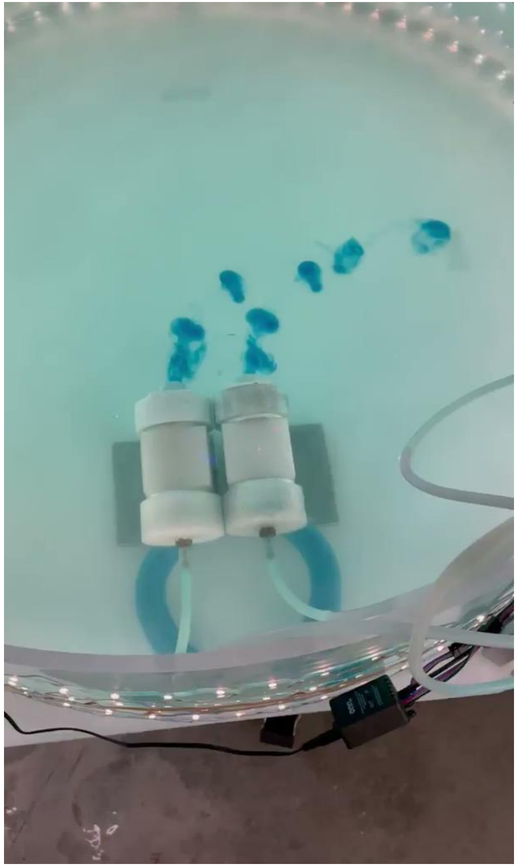
Emerging Trends





Emerging Trends

- Vortex rings are more stable at higher pressures (90-95 psi)
- Setting a higher duration increases the velocity of the rings
- Setting a higher valve increases the velocity of the rings
- Ideal conditions for rotation



Further Study

- Understanding how the individual vortex rings scale to clouds in the atmosphere
- Studying the deformation of vortex rings
- Visualizing and quantifying the mean flow generated by the vortices

Thank you!!

Prezley Content

- Incompressible Fluid Dynamics
  - Mass continuity equation
  - Navier-Stokes momentum equation + terms
  - Stream functions

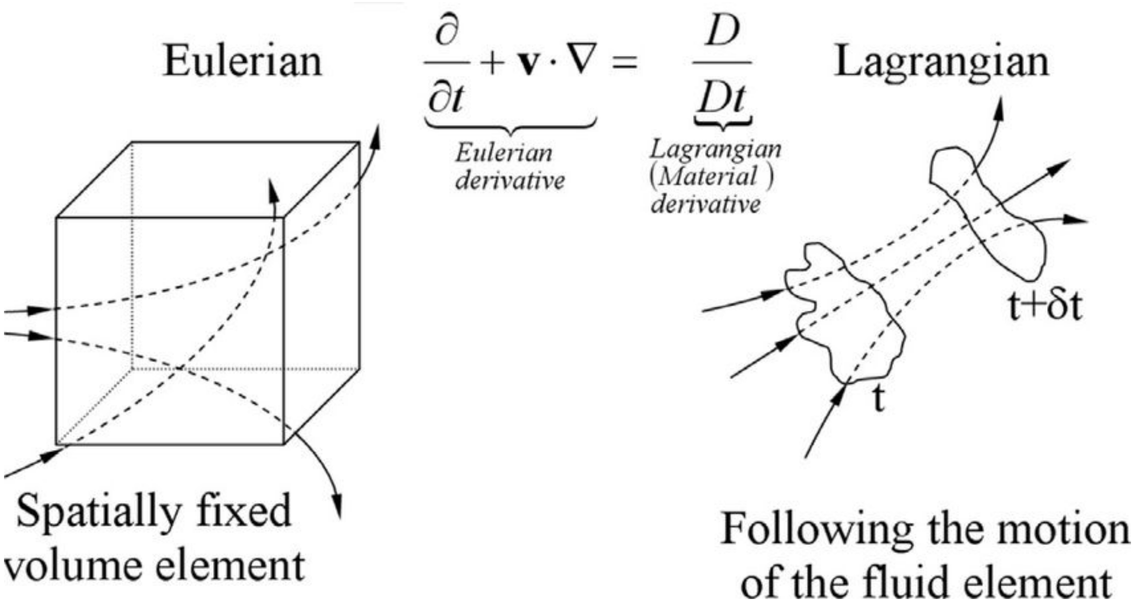
- Rotating Frames of Reference
  - Coriolis Force

- Overview of Igel and Biello's 2020 paper
  - Non-traditional Coriolis terms
  - DoNUT model of convection
  - Coriolis shear force

- Lab Talk

Lagrangian/material view - analysis of fluid motion in terms of positions, momenta of particular fluid elements.

Eulerian/field view - concerns time evolution of the fluid field from a particular frame of reference; how the field evolves in space and time



Incompressible Flow

Mass Continuity Equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$$

Incompressible -density remains constant ($\frac{D\rho}{Dt} = 0$) within a parcel of fluid that moves with the flow, i.e the fluid's velocity field $\vec{u}$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \implies \nabla \cdot \vec{u} = 0$$

-good approximation for fluids with little density variation, where Mach number < 0.3

-$\vec{u}$ can be expressed as the curl of a potential Ψ, $\vec{u} = \nabla \times \Psi$

The Momentum Equation of Fluids

-a PDE describing how the velocity or momentum of a fluid responds to internal and imposed forces

let $\vec{m}(x,y,z,t)$ be the momentum-density field of the fluid
$\implies$ $\vec{m} = \rho\vec{u}$, and the total momentum of a volume of fluid = $\int_V \vec{m} dV$

The rate of change of $\vec{m}$ of a fluid mass is given by the material derivative, and is equal to the forces acting upon it

$$\frac{D}{Dt}\int_V \rho\vec{u} dV = \int_V \vec{F} dV \tag{4}$$

LHS can be rewritten as $\int_V \rho \frac{D\vec{u}}{Dt} dV$,

$$(4) \implies \int_V (\rho\frac{D\vec{u}}{Dt} - \vec{F}) dV = 0$$

The control volume is arbitrary, so the integrand must vanish $\implies \rho\frac{D\vec{u}}{Dt} = \vec{F}$

Expanding the material derivative gives the general momentum equation:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = \frac{\vec{F}}{\rho} \tag{5}$$

Inserting the pressure gradient and viscosity term (contact forces), w/ the assumption that $\nabla \cdot \vec{u} = 0$ gives the incompressible Navier-Stokes equations:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\nabla p + \nu \nabla^2 \vec{u} + \frac{\vec{F}}{\rho}$$

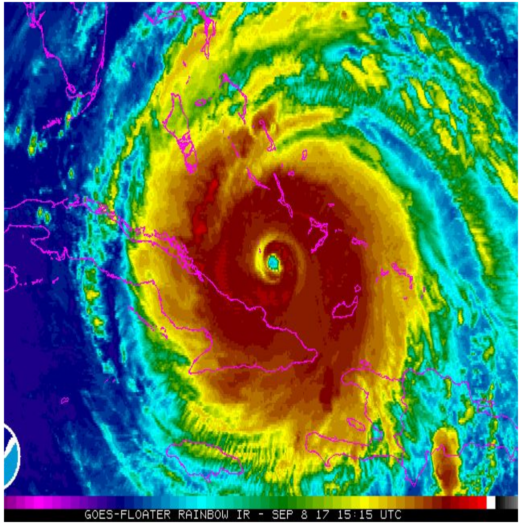
$$\nabla \cdot \vec{u} = 0$$

Vorticity- measures the local rotation of a fluid at some point

Taking the curl of the Navier-Stokes momentum equation, applying several vector identities, and using the assumption that $\nabla \cdot \vec{u} = 0$

$\Longrightarrow$ Vorticity Equation:

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{u} \cdot \nabla)\vec{\omega} - (\vec{\omega} \cdot \nabla)\vec{u} = \nu \nabla^2 \vec{\omega}$$

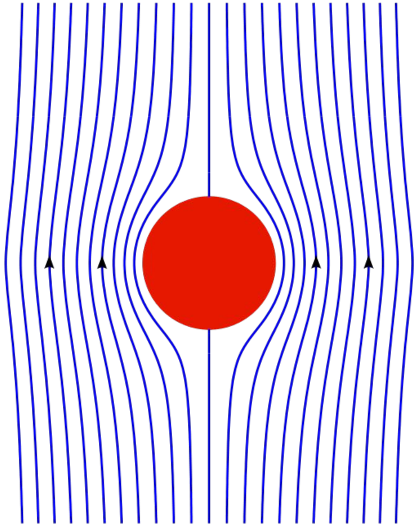


-2nd term on LHS represents vorticity transport by the velocity

-3rd term on LHS represents vortex stretching and rotation

-RHS term accounts for viscous diffusion of the vorticity distribution

Stream Functions



Consider $\vec{u} = u\hat{i} + v\hat{j}$

Define stream function $\psi : u = -\frac{\partial \psi}{\partial y},\ \ v = \frac{\partial \psi}{\partial x}$

$$\implies \vec{u} = -\frac{\partial \psi}{\partial y}\hat{i} + \frac{\partial \psi}{\partial x}\hat{j}, \quad \nabla \psi = \frac{\partial \psi}{\partial x}\hat{i} + \frac{\partial \psi}{\partial y}\hat{j}$$

Note that $\vec{u} \cdot \nabla \psi = 0 \implies \vec{u}$ is parallel to contour lines of constant ψ

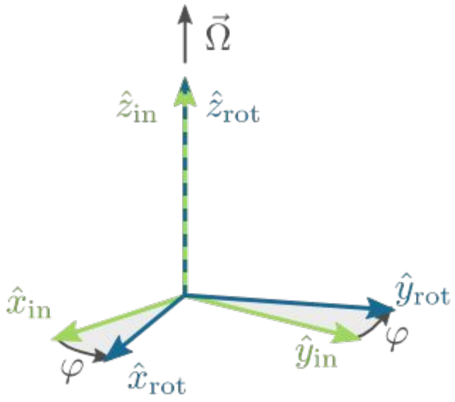
-ψ satisfies continuity of mass (incompressibility): $\nabla \cdot \vec{u} = -\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial x \partial y} = 0$
-ψ is constant along the streamline of the flow
-the difference between ψ at any two points gives the volumetric flow rate through a line connecting the points

$$\text{Vorticity,}\quad \vec{\omega} \equiv \nabla \times \vec{u} = \nabla \times (-\frac{\partial \psi}{\partial y}\hat{i} + \frac{\partial \psi}{\partial x}\hat{j}) = (\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2})\hat{k}$$

$$\vec{\omega} = \nabla^2 \psi$$

Combining $\vec{\omega} = \nabla^2 \psi$ with the vorticity equation gives the Stream function and vorticity formulation of the Navier-Stokes equations

Rotating Reference Frames



let $\vec{\Omega} = \dot{\theta}$, $\vec{v} = \dot{x}$

In a rotating reference frame, the direction of basis vectors change over time

$$\frac{d\hat{i}}{dt} = \vec{\Omega} \times \hat{i}, \quad \frac{d\hat{j}}{dt} = \vec{\Omega} \times \hat{j}, \quad \frac{d\hat{k}}{dt} = \vec{\Omega} \times \hat{k}$$

Consider arbitrary $\vec{u} = u\hat{i} + v\hat{j} + w\hat{k}$

$$\frac{d\vec{u}}{dt} = \left(\frac{\partial u}{\partial t}\hat{i} + \frac{\partial v}{\partial t}\hat{j} + \frac{\partial w}{\partial t}\hat{k}\right) + \left(u\frac{d\hat{i}}{dt} + v\frac{d\hat{j}}{dt} + w\frac{d\hat{k}}{dt}\right) = \left(\frac{\partial u}{\partial t}\hat{i} + \frac{\partial v}{\partial t}\hat{j} + \frac{\partial w}{\partial t}\hat{k}\right) + \left(u(\vec{\Omega} \times \hat{i}) + v(\vec{\Omega} \times \hat{j}) + w(\vec{\Omega} \times \hat{k})\right)$$

$$\implies \left(\frac{d\vec{u}}{dt}\right)_I = \left(\frac{d\vec{u}}{dt}\right)_R + \vec{\Omega} \times \vec{u} \qquad (1)$$

$(\frac{d\vec{u}}{dt})_I$ represents the time derivative of $\vec{u}$ in the stationary reference frame, and $(\frac{d\vec{u}}{dt})_R$ represents the time derivative in the rotating frame

Now consider position vector $\vec{r}$: $(\frac{d\vec{r}}{dt})_I = (\frac{d\vec{r}}{dt})_R + \vec{\Omega} \times \vec{r} \implies \vec{v}_I = \vec{v}_R + \vec{\Omega} \times \vec{r}$

Plug $\vec{v}_I$ into general expression (1) :

$$\implies (\frac{d\vec{v}_I}{dt})_I = (\frac{d\vec{v}_I}{dt})_R + \vec{\Omega} \times \vec{v}_I = (\frac{d}{dt}(\vec{v}_R + \vec{\Omega} \times \vec{r}))_R + \vec{\Omega} \times (\vec{v}_R + \vec{\Omega} \times \vec{r})$$

$$\implies \vec{a}_I = \vec{a}_R + \frac{d}{dt}(\vec{\Omega} + \vec{r}_R) + \vec{\Omega} \times \vec{v}_R + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \ = \ \vec{a}_R + \vec{\Omega} \times \frac{d\vec{r}_R}{dt} + \vec{\Omega} \times \vec{v}_R + \vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

$$\therefore \ \vec{a}_I = \vec{a}_R + (2\vec{\Omega} \times \vec{v}_R) + (\vec{\Omega} \times (\vec{\Omega} \times \vec{r})) \tag{2}$$

Multiply (2) by arbitrary mass, m:

$$\vec{F}_I = \vec{F}_R + 2m(\vec{\Omega} \times \vec{v}_R) + m\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \tag{3}$$

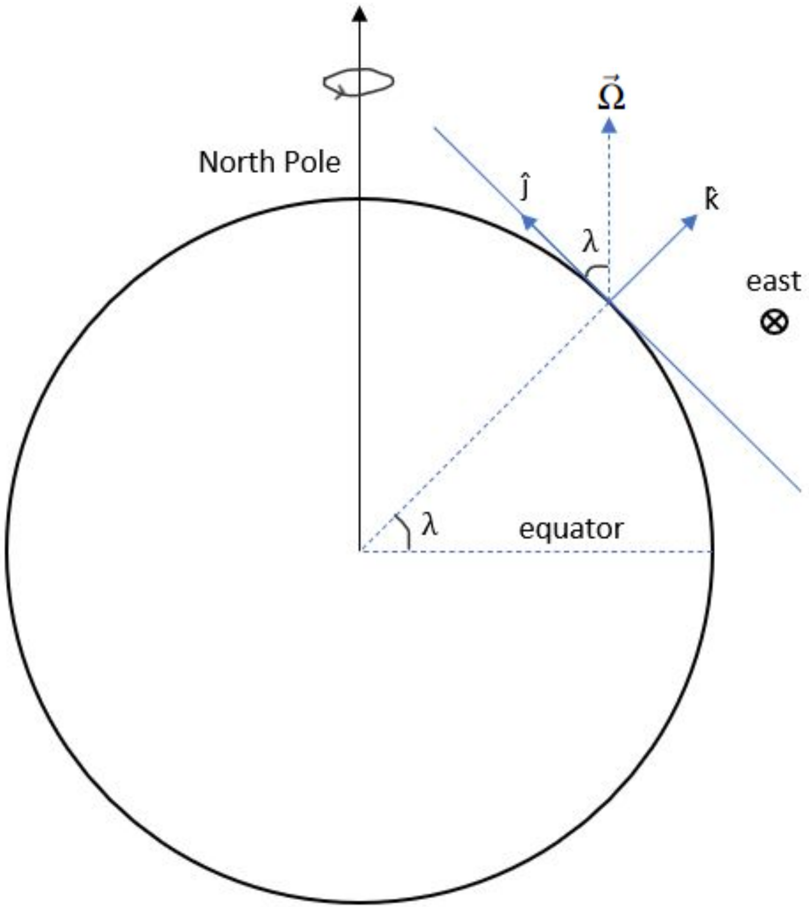
Coriolis Force term: $2m(\vec{\Omega} \times \vec{v}_R)$

Centrifugal Force term: $m\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$

$\hat{j}$ = local north
$\hat{k}$ = local up
λ = latitude
$\vec{\Omega}$ = rotation vector

$$\vec{\Omega} = \Omega_0(cos\lambda\,\hat{j} + sin\lambda\hat{k})$$

Ω_0 = rotation rate



Momentum equation for inviscid fluid in a rotating frame of reference: $\frac{D\vec{u}}{Dt} + \nabla\vec{p} = -2\vec{\Omega} \times \vec{u} + \vec{F}_{ext}$
where $\vec{u} = u\hat{i} + v\hat{j} + w\hat{k}$

$$\implies 2\vec{\Omega} \times \vec{u} = 2\Omega_0[(wcos\lambda - vsin\lambda)\hat{i} + (usin\lambda)\hat{j} - (ucos\lambda)\hat{k}]$$

Momentum equation in component form with non-traditional Coriolis terms underlined,

$$\frac{Du}{Dt} + \frac{\partial p}{\partial x} = 2\Omega_0 vsin\lambda - \underline{2\Omega_0 wcos\lambda}$$

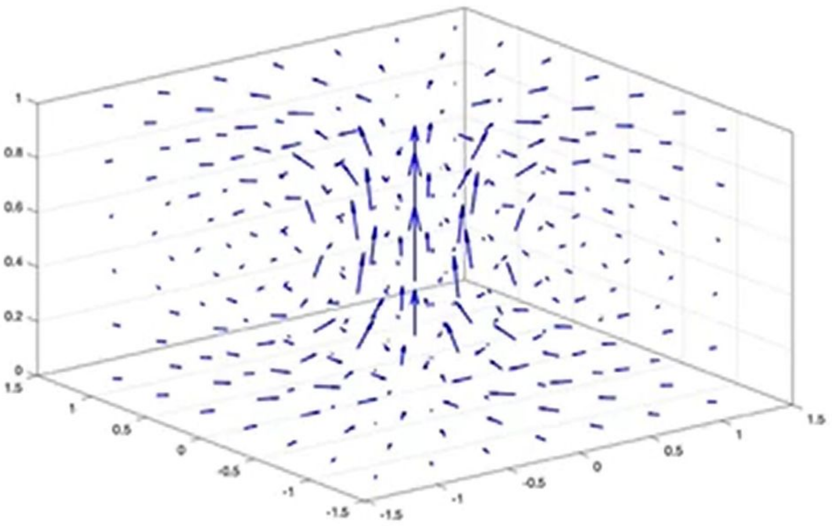
$$\frac{Dv}{Dt} + \frac{\partial p}{\partial y} = -2\Omega_0 usin\lambda$$

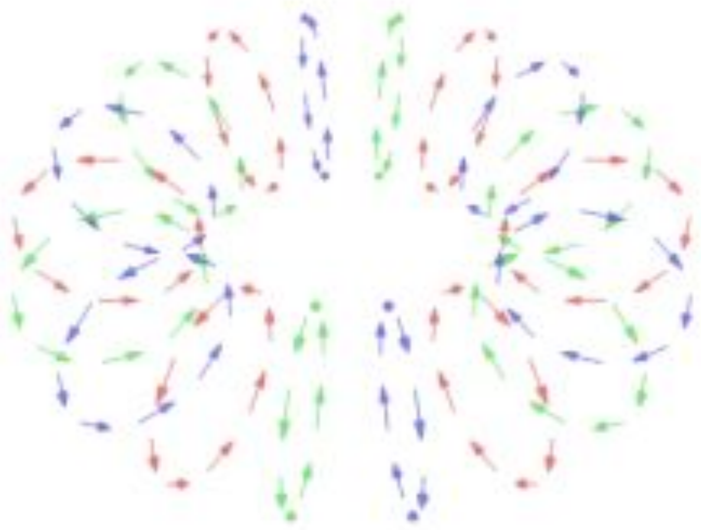
$$\frac{Dw}{Dt} + \frac{\partial p}{\partial z} = \underline{2\Omega_0 ucos\lambda} + F_g$$

Assuming the fluid is incompressible,

$$\nabla \cdot \vec{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

The DoNUT Model of Convection (Igel)





Convection- the transfer of heat between two areas through the movement of fluids

How to determine the effect of NCT?

(Igel and Biello) Define:

$$\vec{F}_{net} = \vec{F}_{tot} - \nabla p_C = -2\vec{\Omega} \times \vec{u} - \nabla p_C$$

$$\text{with } \nabla \cdot \vec{F}_{net} = 0$$

-In general, $\vec{F}_{tot}$ has a net divergence, which would in turn force a divergence on the velocity field

-∇p_C serves to maintain a non-divergent net force, satisfying our incompressibility condition

A Leray projection is then used to diagnose the pressure field and subsequently find a general expression for the net, nondivergent force required to maintain a non-divergent flow $\vec{u}$

$$\mathbf{F}_{\text{net}} = 2\Omega_0 \left\{ \cos(\phi)\nabla^{\perp}[\Psi \sin(\theta)] + \sin(\phi)\frac{\partial \Psi}{\partial z}\hat{\boldsymbol{\theta}} \right\}$$

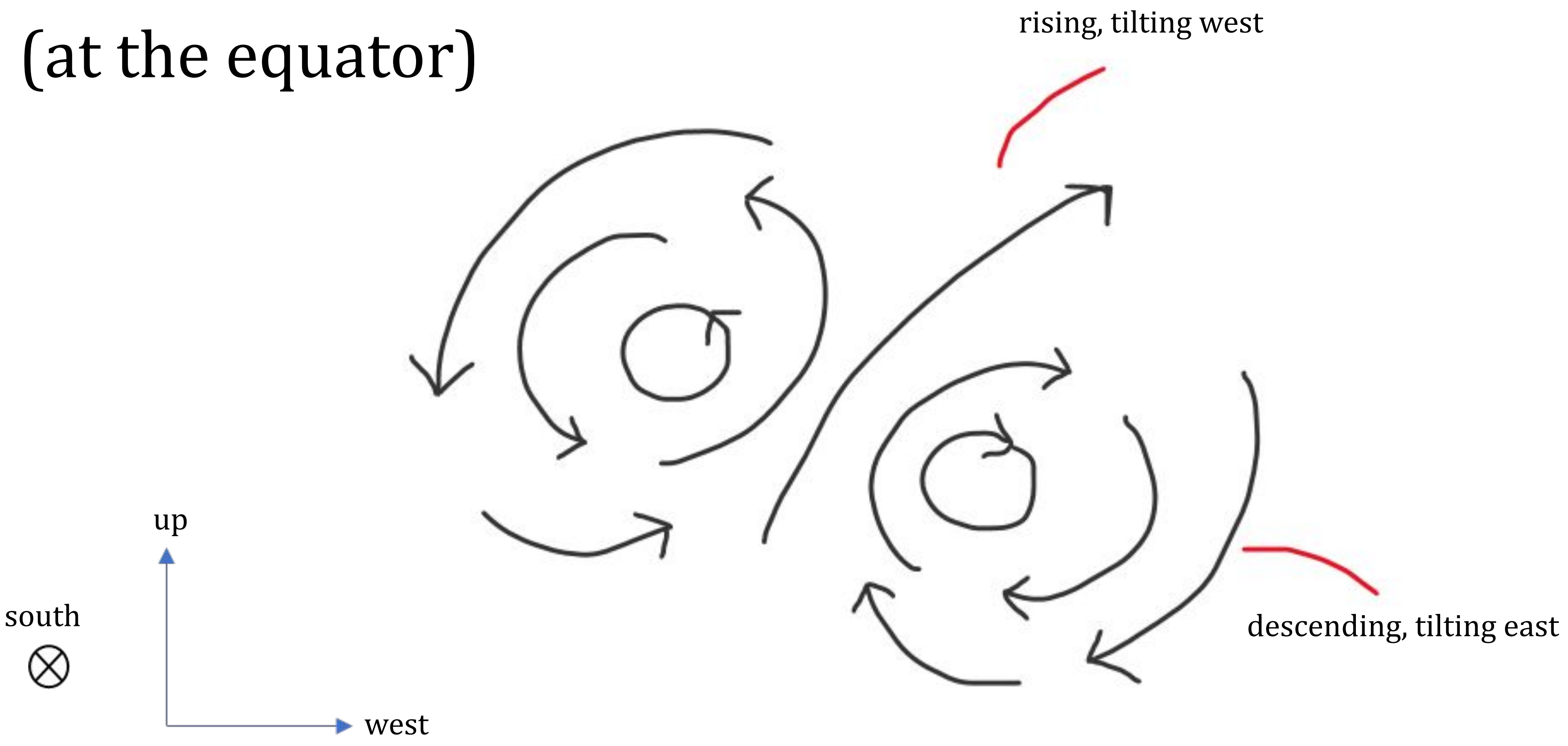
Applying this Leray projection to the DoNUT model with a prescribed vector potential, Ψ, associated with the poloidal flow

$$\mathbf{F}_{\text{CS}}(0,z) = -2\Omega_0 \cos(\phi)\frac{\partial G}{\partial y}\hat{\mathbf{i}} = -\Omega_0 \cos(\phi)\, w(z)\,\hat{\mathbf{i}}$$

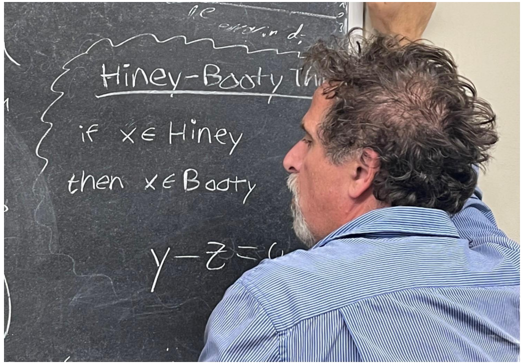
-$\vec{F}_{CS}$ points **westward** in the **ascending** part of the convective flow

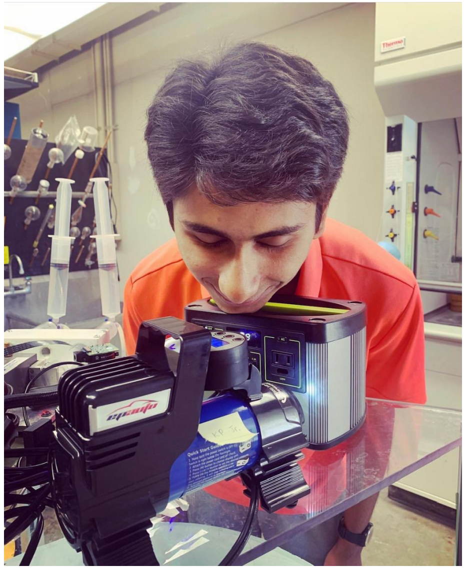
-$\vec{F}_{CS}$ is proportional to the strength of vertical velocity along central axis

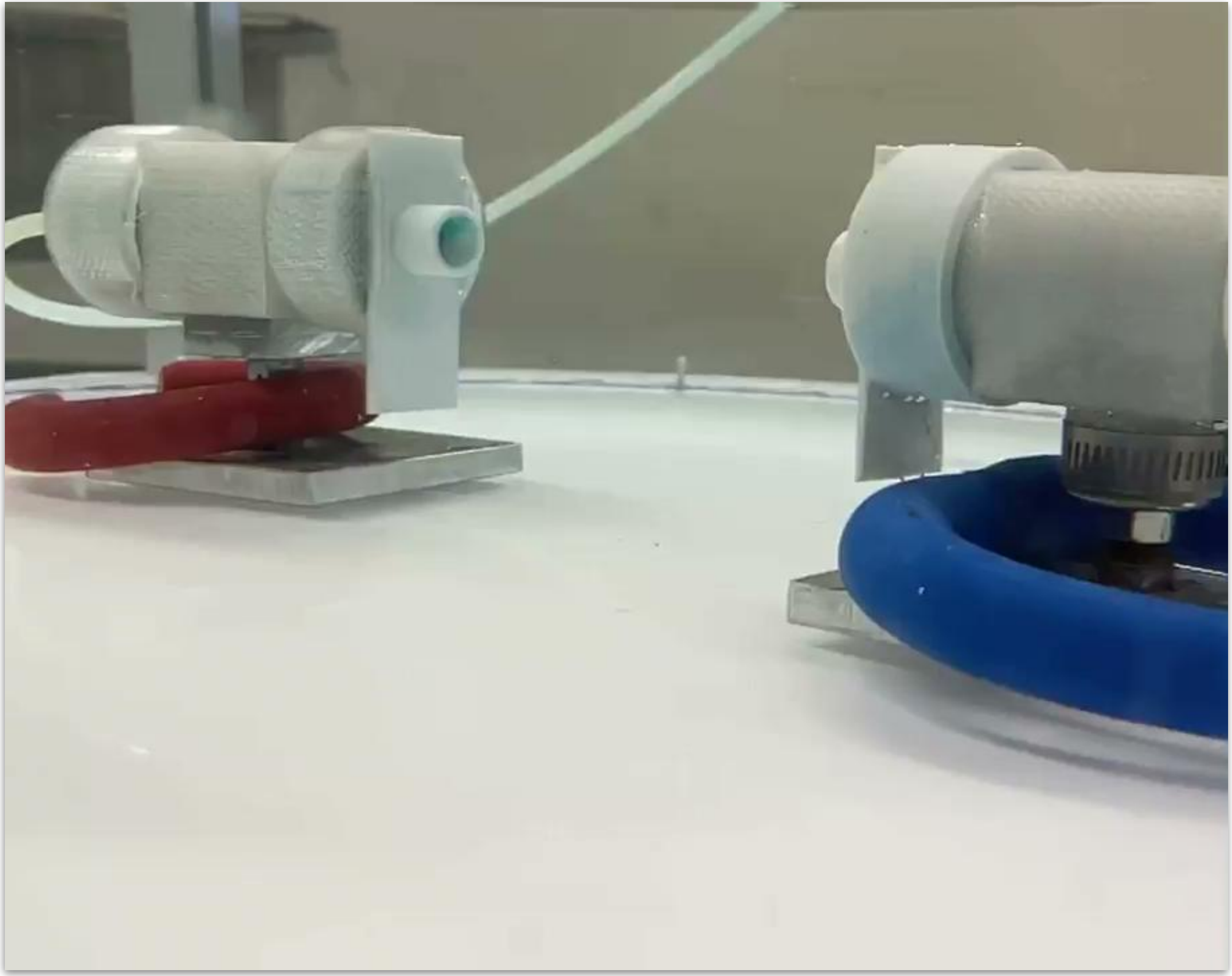
Poloidal Convection (at the equator)



Lab!







Slide of Appreciation