Lie algebra representation theory, plane partitions, and Clebsch-Gordan over $\mathbb{Z}$

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Abstract. We examine a method for counting plane partitions which uses the representation theory of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. Motivated by the desire to use this method to study the Kasteleyn cokernel, we construct a possible $\mathbb{Z}$-analogue of Clebsch-Gordan decomposition, and discuss its properties, as well as plans to study its structure.

Contents

1. Preliminaries and motivation

The counting of plane partitions has been studied in great depth. Plane partitions and several of their symmetry classes were first considered by MacMahon [7]; the remaining symmetry classes as well as formulas for their enumeration were identified by Stanley [10] and Robbins [8], and at present, all ten of these formulae have already been successfully been proven. Nevertheless, the methods used to solve these problems can themselves be studied, and so they also serve as sources of interesting new questions.

In particular, this paper examines the Kasteleyn cokernel, the cardinality of which is the solution to the counting problem via the methods of Kasteleyn [3] and Percus [9]. A natural question to ask is, "Does the abelian group structure of this cokernel have any fundamental meaning? Is there something deeper underlying this object?"

We seek to pursue this question by investigating a connection between this cokernel and the representation theory of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. This pursuit eventually leads us to a more general question of representation theory, as we eventually pursue a $\mathbb{Z}$-analogue for the Clebsch-Gordan decomposition of tensor products of irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$.

In this section, we discuss important preliminary definitions and notation, and introduce the aforementioned methods of counting plane partitions.

1.1. Plane partitions

Definition 1.1. A *plane partition* in an $a \times b \times c$ sized box is a stable stack of unit cubes: a collection of cubes $I^3 \subseteq [0,a] \times [0,b] \times [0,c] \subseteq \mathbb{R}^3$ which does not fall over when pushed towards the origin (i.e. a lower set of these cubes).

Equivalently, we may define a plane partition to be an $a \times b$ matrix with nonnegative entries $\pi_{i,j} \leq c$, such that $\pi_{i,j}$ are nonincreasing across indices i and j. This is a histogram version of the previous definition, in which each entry $\pi_{i,j}$ corresponds to a tower of $\pi_{i,j}$ cubes. For instance, this is the matrix corresponding to the plane partition in Figure 1 below:

$$\begin{bmatrix} 4 & 3 & 1 \\ 2 & 0 & 0 \end{bmatrix}.$$

Now let the set of $a \times b \times c$ plane partitions be $P(a,b,c)$, or P when the dimensions are arbitrary. The original counting problem was to find $|P|$; one can also consider symmetry classes of these plane partitions (i.e. subsets of P invariant under certain symmetry relations), and count those as well; these values are already known.

In particular, one can find $|P|$ by translating plane partitions into the language of graph theory.

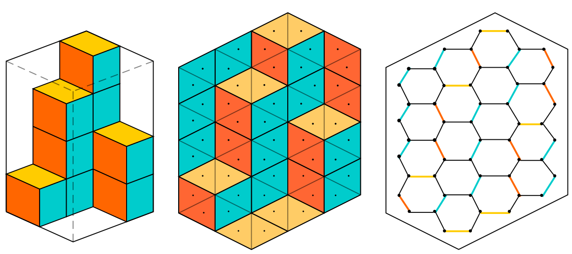


Figure 1. A $2 \times 3 \times 4$ plane partition, its lozenge tiling, and its perfectly matched graph.

Consider the semiregular hexagon of side lengths a, b, c (we define a *semiregular hexagon* to be a hexagon whose opposite sides share the same length). These hexagons can be tiled by lozenges (i.e. $60°$ unit rhombi, sometimes called *calisson*). Then $P(a,b,c)$ is in bijection with the set of lozenge tilings of the $a \times b \times c$ semiregular hexagon.

One can then correspond a lozenge tiling to a perfect matching of a planar bipartite graph; indeed, let each lozenge be divided into two equilateral triangles, which point either east or west. Let each triangle define a vertex of our graph, and let edges be between adjacent triangles. The perfect matching is formed by the edges which comprise lozenges in the tiling.

Then the set of lozenge tilings is in bijection with the set of perfect matchings. Therefore, by composing the two bijections, we can count plane partitions by counting perfect matchings of the aforementioned planar bipartite graph [5]. I have depicted these bijections visually in Figure 1 above.

1.2. The Kasteleyn cokernel

It is well-known that one can compute the number of perfect matchings of a bipartite graph by computing the permanent of its bipartite adjacency matrix. However, the permanent is difficult to compute, and so we wish to "convert" it into a determinant, which we can do by assigning negative signs to certain entries in the matrix. The following rule by Percus [9] is called the *permanent-determinant method* (or *Kasteleyn-Percus method*).

Theorem 1.1. The edges of a simple, planar, bipartite graph G with bipartite adjacency matrix M can be assigned signs $+, -$ such that $\det M' = \operatorname{perm} M$, where M' is the signed matrix, and $\operatorname{perm} M$ is the permanent of M. In particular, the rule is as follows: each face of G will contain an odd number of $-$ signs if and only if the face has $4k$ sides.

Thus, the determinant of this altered matrix (also known as its *Kasteleyn-Percus* matrix) is the solution to our counting problem.

Definition 1.2. The *Kasteleyn cokernel* A of a simple, planar, bipartite graph is the cokernel of its Kasteleyn-Percus matrix M.

Recall that the order of this cokernel is $\det M$, because the Smith normal form of M is a diagonal matrix with diagonal entries $\{a_i\}_i$, so that $\operatorname{coker}(M) \cong \bigoplus_i \mathbb{Z}/a_i$, and $\prod_i a_i = \det M$. Therefore, $|A|$ is the desired number of plane partitions.

Additionally, note that when considering certain symmetry classes, we find that our graphs are not always bipartite, in which case there exists a generalization of the above method (the *Hafnian-Pfaffian* or *Kasteleyn method* [3]). Nevertheless, for the purposes of this paper, we merely examine the general class of plane partitions, for which the determinant already equals the permanent by the permanent-determinant method.

The structure of the Kasteleyn cokernel appears to be nontrivial. For instance, the number of domino tilings of the Aztec diamond of order n is $2^{n(n+1)/2}$, and its respective Kasteleyn cokernel is $\mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/8 \oplus \cdots \oplus \mathbb{Z}/2^n$. However, it is unknown what determines this structure.

Thus we may seek to connect it to other methods of finding $|P|$.

1.3. The $\mathfrak{sl}(2, \mathbb{C})$ method

In this section, we will see how plane partitions arise in the representation theory of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$, particularly through tensor products of certain irreps. This method was first used by Kuperberg [4] to enumerate four symmetry classes of plane partitions at once. Let us now recall some basic facts about $\mathfrak{sl}(2, \mathbb{C})$ and its representation theory.

Definition 1.3. The complex special linear Lie algebra of order 2, denoted $\mathfrak{sl}(2, \mathbb{C})$, is the vector space defined to be the complex span of the matrices

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

which is made further into a non-associative algebra by the Lie bracket $[A, B] = AB - BA$ for $A, B \in \mathfrak{sl}(2, \mathbb{C})$. In other words, it consists of of 2×2 traceless matrices with entries in $\mathbb{C}$, together with the aforementioned Lie bracket.

Theorem 1.2. Let (V, α) be a finite dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$. Then the following two facts are true:

(1) V decomposes (up to reordering) into a direct sum of irreducible subrepresentations.
(2) The irreducible representations V_n of $\mathfrak{sl}(2, \mathbb{C})$ are classified by $\mathbb{N}$ up to isomorphism, where $\dim V_n = n + 1$. (For more information on $\mathfrak{sl}(2, \mathbb{C})$, see Section 4.6 of reference [1].)

Up to isomorphism, V_n is the $(n + 1)$-dimensional space of homogeneous degree n polynomials in $\mathbb{Z}[x, y]$, with representation $\alpha_n : \mathfrak{sl}(2, \mathbb{C}) \to \operatorname{Aut}(V_n)$. The action of $\mathfrak{sl}(2, \mathbb{C})$ is defined by

$$\alpha_n(X) = x\frac{\partial}{\partial y}, \quad \alpha_n(Y) = y\frac{\partial}{\partial x}, \quad \alpha_n(H) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

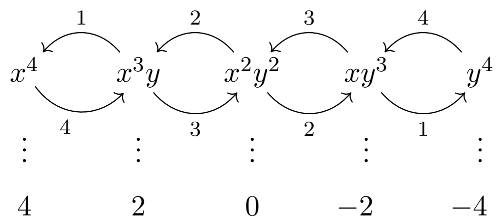
We can also write these actions on a basis vector $x^{n-k}y^k$:

$$\begin{aligned} \alpha_n(X)x^{n-k}y^k &= kx^{n-k+1}y^{k-1} \\ \alpha_n(Y)x^{n-k}y^k &= (n-k)x^{n-k-1}y^{k+1} \\ \alpha_n(H)x^{n-k}y^k &= (n-2k)x^{n-k}y^k. \end{aligned}$$

Then each basis vector $x^{n-k}y^k$ is an eigenvector under the H action with eigenvalue $n-2k$; thus, they are uniquely identifable by their eigenvalue, and so we can denote them by e_i, where $i \in \{-n, -n+2, \ldots, n-2, n\}$. It is helpful to rewrite the actions in terms of this new notation:

$$\alpha_n(X)e_i = \tfrac{n-i}{2}e_{i+2}, \quad \alpha_n(Y)e_i = \tfrac{n+i}{2}e_{i-2}, \quad \alpha_n(H)e_i = ie_i.$$

In the diagram below, we display the action of $\mathfrak{sl}(2,\mathbb{C})$ on V_4.



The leftwards-pointing sequence of arrows represents the action of X (with the arrow labels being the coefficient of multiplication); the rightwards-pointing sequence of arrows represents the action of Y; and the numbers at the bottom are the eigenvalues corresponding to the action of H.

Definition 1.4. If (V,α) is a representation of $\mathfrak{sl}(2,\mathbb{C})$, and we have $v \in V$ such that $\alpha(H)v = \lambda v$, then v is called a *weight vector*, and λ is its *weight*. Then eigenspaces $V|_\lambda$ are called *weight spaces*.

Furthermore, recall that the action of an element X of an arbitrary Lie algebra on a tensor product of representations $\alpha \otimes \beta$ is as follows:

$$(\alpha \otimes \beta)(X) = \alpha(X) \otimes \mathrm{id} + \mathrm{id} \otimes \beta(X).$$

Thus, on a tensor product $V_n \otimes V_k$ with bases $\{e_i\}, \{f_j\}$ respectively, we have

$$(\alpha_n \otimes \alpha_k)(H)(e_i \otimes f_j) = (ie_i) \otimes f_j + e_i \otimes (jf_j) = (i+j)(e_i \otimes f_j).$$

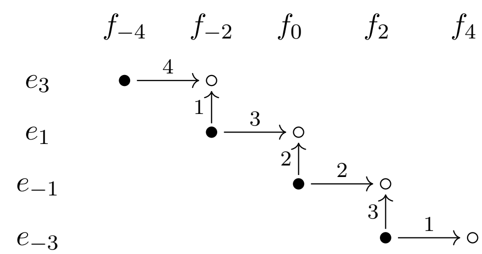
In other words, the basis vectors of a tensor product of representations are also weight vectors, and their weights are the sums of the weights of their constituents.

Now, given a tensor product $V := V_n \otimes V_k$, let us consider the the action of X on the weightspace $V|_{-1}$; indeed, by the action defined above, we have $\alpha(X)|_{-1} : V|_{-1} \to V|_{+1}$.

Example 1.1. Consider $V := V_3 \otimes V_4$. Let us denote the weight vectors of V_3 by $\{e_{-3}, e_{-1}, e_1, e_3\}$ and the the weight vectors of V_4 by $\{f_{-4}, f_{-2}, f_0, f_2, f_4\}$; now, we will compute $\alpha(X)|_{-1}$ on $V|_{-1}$.

In the diagram to the right, we have visually depicted $\alpha(X)|_{-1} : V|_{-1} \to V|_{+1}$ in a table of basis vectors. The black bullets depict vectors of weight -1, the white bullets depict vectors of weight $+1$, and the arrows depict the action of X. For instance, we can calculate

$$(\alpha_3 \otimes \alpha_4)(X)(e_1 \otimes f_{-2}) = e_3 \otimes f_{-2} + 3e_1 \otimes f_0.$$



Thus, at the black bullet depicting $e_1 \otimes f_{-2}$, there is one arrow labelled "1" pointing up to $e_3 \otimes f_{-2}$, and one arrow labelled "3" pointing right to $e_1 \otimes f_0$.

In fact, the same procedure can be done for three-fold tensor products. In particular, if we denote $\hat{a} = b + c - 1$, $\hat{b} = a + c - 1$, and $\hat{c} = a + b - 1$, we have that the matrix $\alpha(X)|_{-1}$ in the product $V_{\hat{c}} \otimes V_{\hat{b}} \otimes V_{\hat{a}}$ is a weighted Kasteleyn-Percus matrix for an $a \times b \times c$ plane partition.

In Figure 2 below, the basis vectors of the three-fold product $V_4 \otimes V_4 \otimes V_5$ now fill a three-dimensional table. Again, black bullets depict the -1 weight space filled-in with magenta, white bullets depict the $+1$ weight space filled-in with cyan, and edges depict the action of X. We see that this structure exactly takes the familiar form of a bipartite graph for a $2 \times 2 \times 3$ plane partition.

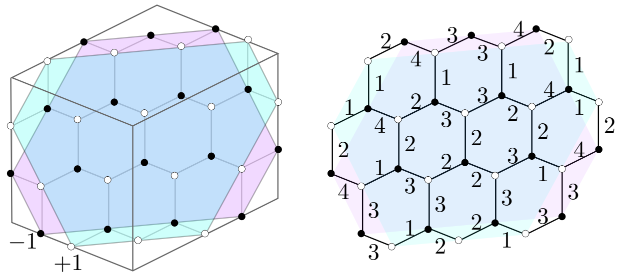


FIGURE 2. The weighted bipartite graph of $2 \times 2 \times 3$ plane partition, seen in the action $\alpha(X)|_{-1}$ on $V_4 \otimes V_4 \otimes V_5$.

The associated graph is *Kasteleyn-flat* (i.e. opposite edges have the same weight), and thus $\det \alpha(X)|_{-1}$ is the determinant of its unweighted version times a known factor. Dividing by this factor, we obtain our original $|A|$ [4]. However, taking the Smith normal form of this weighted adjacency matrix to obtain its cokernel gives us a large pre-quotient of our original Kasteleyn.

If we wished, it would be simple to change the basis of our representations such that the action of X has a coefficient of 1, so that the determinant would be exactly $|A|$, and the matrix would exactly be the Kasteleyn-Percus matrix which yields us the Kasteleyn cokernel. More explicitly, we can do as follows:

Proposition 1.1. Let $\{e_{n-2k}\}$ denote our aforementioned basis. Then define

$$\hat{e}_{n-2k} = \frac{n!}{k!} \cdot e_{n-2k}.$$

The set $\{\hat{e}_{n-2k}\}$ is a basis which normalizes the action of X.

Proof. The transformation is clearly linear. Furthermore, the action of X has a coefficient of 1:

$$\begin{aligned}\alpha(X)\hat{e}_{n-2k} &= \frac{n!}{k!}\alpha(X)e_{n-2k} \\ &= \frac{n!}{k!} \cdot k e_{n-2(k-1)} \\ &= \frac{n!}{(k-1)!} \cdot e_{n-2(k-1)} \\ &= \hat{e}_{n-2k+2}\end{aligned}$$

We can also calculate the Y-action:

$$\begin{aligned}\alpha(Y)\hat{e}_{n-2k} &= \frac{n!}{k!}\alpha(Y)e_{n-2k} \\ &= \frac{n!}{k!} \cdot (n-k)e_{n-2(k+1)} \\ &= \frac{n!}{k!} \cdot (n-k) \cdot \frac{(k+1)!}{n!}\hat{e}_{n-2(k+1)} \\ &= (n-k)(k+1)\hat{e}_{n-2(k+1)}\end{aligned}$$

Since this change of basis is just a shift by scalar factor, the weights and weight vectors do not change. □

However, returning to the original calculation, we still require one more important theorem in order to compute the determinant:

Theorem 1.3. *(Clebsch-Gordan)* Let V_n and V_k, $n \geq k$, be two irreps of $\mathfrak{sl}(2, \mathbb{C})$. Then we have the following decomposition:

$$V_n \otimes V_k \cong V_{n+k} \oplus V_{n+k-2} \oplus V_{n+k-4} \oplus \cdots \oplus V_{n-k}.$$

This decomposition effectively diagonalizes our aforementioned weighted Kasteleyn-Percus matrix, allowing us to enumerate $|P|$. Furthermore, this process can be extended to the quantum group generalization $U_q(\mathfrak{sl}_2)$ to compute the q-enumerated case, as well [4]. (For more information on $U_q(\mathfrak{sl}_2)$, see Section V of reference [2]).

As Clebsch-Gordan is equally valid over $\mathbb{Q}$, we can use $\mathfrak{sl}(2,\mathbb{Q})$ in place of its complex counterpart. However, the proof of Clebsch-Gordan requires properties of $\mathbb{Q}$ as a field. If we seek to connect this method to the Kasteleyn cokernel, which is meaningful only over $\mathbb{Z}$, we would need to construct a $\mathbb{Z}$-analogue of Clebsch-Gordan.

In order to do so, we will define analogues of our irreducible representations, and realize them as submodules of the tensor product in a way that preserves the desired connection to the Kasteleyn cokernel. In the next section, we find that this procedure results in another cokernel to study.

2. An analogue over the integers

First, we begin with a generalization of our Lie algebra:

Definition 2.1. The special linear Lie algebra of order 2 over the integers, denoted $\mathfrak{sl}(2,\mathbb{Z})$, is the $\mathbb{Z}$-module defined to be the integer span of the matrices

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

which is made further into a non-associative algebra over the integers by the Lie bracket $[A,B] = AB - BA$ for $A, B \in \mathfrak{sl}(2,\mathbb{Z})$. In other words, it consists of of 2×2 traceless matrices with entries in $\mathbb{Z}$, together with the aforementioned Lie bracket.

We can similarly define a $\mathbb{Z}$-analogue of V_n to be an $(n+1)$-dimensional free $\mathbb{Z}$-module with basis $\{e_i\}_i$, where $i \in \{-n, -n+2, \ldots, n-2, n\}$, together with actions

$$\alpha_n(X)e_i = \tfrac{n-i}{2} e_{i+2}, \quad \alpha_n(Y)e_i = \tfrac{n+i}{2} e_{i-2}, \quad \alpha_n(H)e_i = ie_i.$$

Note that $n \pm i$ is always even, so the actions are closed over the integers. Then, since the actions are preserved, and the change of basis described in Proposition 1.1 is also closed over the integers, there is reason to hope we can obtain our Kasteleyn cokernel.

Nevertheless, these V_n are no longer irreducible; more importantly, their tensor products no longer decompose via Clebsch-Gordan as above. Yet since the actions are closed over the integers, indeed the weights and weight vectors are preserved. Thus, one can hope to proceed with the standard way of decomposing a representation of $\mathfrak{sl}(2,\mathbb{Q})$, i.e. running through the action of Y on the highest weight vector to create a subrepresentation. We base the following construction on this procedure.

2.1. Construction

There are two important factors to consider: first, we want our decomposition to be composed of subrepresentations, i.e. they must be both submodules and closed under the action of $\mathfrak{sl}(2,\mathbb{Z})$.

Furthermore, we want components of our decomposition to be complemented, so that our decomposition can actually exist; this fact is true for any complex semisimple Lie algebra, but is not always true over the integers. For instance, we have the following simple motivating example:

Example 2.1. Consider the integer lattice $\mathbb{Z}^2$ with basis $a = (0,1), b = (1,0)$, and consider the submodule generated by $2a$. Then there does not exist a complement to $\langle 2a \rangle$, i.e. a submodule N such that $\mathbb{Z}^2 = \langle 2a \rangle \oplus N$ (where we take the internal direct sum, so we require $\langle 2a \rangle \cap N = \varnothing$). If there did exist such an N, then there would exist some $x \in N$ such that $2ka + x = a$, so $x = (2k-1)a$; then $2x \in \langle 2a \rangle$, so $\langle 2a \rangle \cap N \neq \varnothing$, a contradiction.

It is not uncommon that we obtain a non-primitive element (recall that a *primitive element* of a lattice is one that is not an integer multiple of another element) when we compute the orbit of the Y-action on the highest weight vector. Thus, we wish to design a module which could have a complement, but is still in "the same direction" as before, i.e. $\mathfrak{sl}(2,\mathbb{Z})$ actions are preserved.

Definition 2.2. Let $M \subseteq \mathbb{Z}^n$ be a finitely-generated $\mathbb{Z}$-module. Then we define the *rational closure* $M^{\mathbb{Q}}$ of M in $\mathbb{Z}^n$ to be

$$M^{\mathbb{Q}} := (M \otimes_{\mathbb{Z}} \mathbb{Q}) \cap \mathbb{Z}^n,$$

or equivalently, the kernel of the following sequence of maps:

$$\mathbb{Z}^n \to \mathbb{Z}^n/M \to F(\mathbb{Z}^n/M)$$

where the first map is the canonical projection to the quotient, and the second map is the canonical quotient by torsion submodule.

In the case of our Clebsch-Gordan construction, the rational closure eliminates the issue created by non-primitive elements. First, we have the following weaker result:

Proposition 2.1. Let $M \subseteq \mathbb{Z}^n$ be a finitely-generated $\mathbb{Z}$-module. Then $M^{\mathbb{Q}}$ is complemented in $\mathbb{Z}^n$, i.e. there exists a $\mathbb{Z}$-submodule N of $\mathbb{Z}^n$ such that $\mathbb{Z}^n = M^{\mathbb{Q}} \oplus N$.

Proof. Let $\{x_1, \ldots, x_m\} \subseteq \mathbb{Z}^n$ be the generators of M, and consider the matrix $[x_1 \cdots x_m]$. Taking the Smith normal form of this matrix yields us a change of basis in which M is *rectilinear* (i.e. its generators are each proportional to a standard basis vector of $\mathbb{Z}^n$). Then taking the rational closure of a rectilinear basis yields a subset of the standard basis, and hence $\mathbb{Z}^n/M^{\mathbb{Q}}$ is generated by the remaining standard basis vectors. Thus, we have the splitting $\mathbb{Z}^n = M^{\mathbb{Q}} \oplus (\mathbb{Z}^n/M^{\mathbb{Q}})$. □

We will use this proposition to demonstrate that each component of our $\mathbb{Z}$-analogue decomposition is complemented in $V_n \otimes V_k$.

Let us now at last state the construction of our analogue.

Definition 2.3. Let $n \geq k$, and let V_n, V_k be the previously defined $\mathbb{Z}$-analogues of their respective $\mathfrak{sl}(2,\mathbb{Q})$ irreducible representations. Let the v be highest weight vector of $V_n \otimes V_k \cong \mathbb{Z}^{(n+1)(k+1)}$, and consider its (linear) *Y-orbit*:

$$W_{n+k} := \langle \{\alpha(Y^n)v \quad \text{s.t.} \quad n \in \mathbb{N}\} \rangle.$$

Next, take the rational closure $W^{\mathbb{Q}}_{n+k}$, and find the next highest weight vector not in $W^{\mathbb{Q}}_{n+k}$, and whose Y-orbit W_{n+k-2} is closed under the $\mathfrak{sl}(2,\mathbb{Q})$ action. Take its rational closure, and repeat this process until we reach the last component, $W^{\mathbb{Q}}_{n-k}$. Then define our *$\mathbb{Z}$-Clebsch-Gordan decomposition (ZCG)* to be:

$$W^{\mathbb{Q}}_{n+k} \oplus W^{\mathbb{Q}}_{n+k-2} \oplus W^{\mathbb{Q}}_{n+k-4} \oplus \cdots \oplus W^{\mathbb{Q}}_{n-k}.$$

Proposition 2.2. The following statements are true:

1. The $W^{\mathbb{Q}}_i$ are generated by orthogonal elements.
2. The $W^{\mathbb{Q}}_i$ exist, and are representations of $\mathfrak{sl}(2,\mathbb{Z})$; i.e., we can find a next-highest weight vector whose Y-orbit is closed under the action of $\mathfrak{sl}(2,\mathbb{Z})$ and not contained in previous components.
3. Each $W^{\mathbb{Q}}_i$ is complemented in $\mathbb{Z}^{(n+1)(k+1)}$.

Proof.

(1) Given a weight vector v of weight λ in $V_n \otimes V_k$, we have that $v = \sum_i a_i v_{\lambda,i}$, where $v_{\lambda,i}$ are the basis elements (simple tensors) of weight λ. Then

$$\alpha(Y)v = \sum_i a_i \alpha(Y) v_{\lambda,i} = \sum_j b_j v_{\lambda-2,j}$$

will thus be a weight vector of weight $\lambda - 2$. Indeed, as their coefficients are only nonzero in different basis vectors, they are orthogonal. Similarly, since all generators of the orbit have different weights, they are all pairwise orthogonal. Note that taking the rational closure of the Y-orbit is equivalent to taking each generator to its primitive form (i.e. via dividing by the gcd of its entries), because the generators exist in different weight spaces.

(2) Within $V_n \otimes V_k \otimes \mathbb{Q}$, we have that $W_i^{\mathbb{Q}}$ is the intersection of submodules $W_i \otimes \mathbb{Q}$ and $V_n \otimes V_k$, representations of $\mathfrak{sl}(2,\mathbb{Z})$; thus, $W_i^{\mathbb{Q}}$ is also a representation of $\mathfrak{sl}(2,\mathbb{Z})$. Nevertheless, we can also provide a more constructive proof of this fact, which clarifies the computation and concrete structure of these components.

We will find that $W_i^{\mathbb{Q}}$ are representations of $\mathfrak{sl}(2,\mathbb{Z})$ by proving their existence. The $W_i^{\mathbb{Q}}$ are already closed under the actions of Y and H. We need only to choose "next highest weight" vector such that the module is closed under the action of X.

First, the rational closure of a Y-orbit generated by a weight vector v is closed under the action of X if and only if $\alpha(X)v = 0$. If $\alpha(X)v = 0$, then since the actions of X and Y are inverses by a scalar factor (which are normalized in the rational closure), the Y-orbit of v is closed under X's action. Likewise, if Y-orbit of v is closed under X's action, then automatically v must vanish under the X-action, or else $\alpha(X)v$ would not be in the Y-orbit.

Hence, let the previous component be $W_{N+2}^{\mathbb{Q}}$, i.e. the Y-orbit of a vector of weight $N+2$. Now, we define our next highest weight vector v to be the generator of $\ker \alpha(X)|_N$; so, it is sufficient to show that v exists. Moreover, v is not contained in $W_i^{\mathbb{Q}}$ for $i > N$, because by (1), all elements of weight N in $W_i^{\mathbb{Q}}$ for $i > N$ do not vanish under $\alpha(X)$.

For the purposes of this proof, let us define another notation for the action of $(2,\mathbb{Z})$: let V_n have basis $\{e_0, \ldots, e_n\}$ where $e_i = x^{n-i}y^i$; then we have the following actions:

$$\alpha(X)e_i = ie_{i-1}, \quad \alpha(Y)e_i = (n-i)e_{i+1}, \quad \alpha(H)e_i = (n-2i)e_i.$$

Thus, we have the following action of X on $V_n \otimes V_k$ (with basis $\{e_i \otimes f_j\}$):

$$(\alpha_n \otimes \alpha_k)(X)(e_i \otimes f_j) = ie_{i-1} \otimes f_j + je_i \otimes f_{j-1}.$$

We desire a collection of coefficients $a_{i,j}$ such that the following is true:

$$\alpha(X) \sum_{i+j=N} a_{i,j} e_i \otimes f_j = \sum_{i+j=N} a_{i,j}(ie_{i-1} \otimes f_j + je_i \otimes f_{j-1}) = 0.$$

Because simple tensors are linearly independent, this corresponds to the system

$$a_{i,j} i = a_{i-1,j+1}(j+1)$$
$$a_{i,j} j = a_{i+1,j-1}(i+1)$$

for all $i + j = N$, which can be solved for the coefficients with one degree of freedom. Thus, we let v be the primitive element satisfying these conditions.

(3) As the $W_i^{\mathbb{Q}}$ are submodules of $V_n \otimes V_k$, they are complemented by Proposition 2.1. ☐

As a final demonstration of this construction, we will compute an example in full.

Example 2.2. For the purposes of this example, let us define another notation for the action of $\mathfrak{sl}(2,\mathbb{Z})$: let V_n have basis $\{e_0,\dots,e_n\}$ where $e_i = x^i y^{n-i}$; then we have the following actions:

$$\alpha(X)e_i = (n-i)e_{i+1}, \quad \alpha(Y)e_i = ie_{i-1}, \quad \alpha(H)e_i = (2i-n)e_i.$$

Consider the product $V_2 \otimes V_1$ with basis $\{e_i \otimes f_j\}$, which will have ZCG

$$W_3^{\mathbb{Q}} \oplus W_1^{\mathbb{Q}}.$$

The highest weight vector is $e_2 \otimes f_1$. Then, we obtain the Y-orbit generators:

$$\begin{aligned} \text{id}: &\quad e_2 \otimes f_1 \\ \alpha(Y): &\quad 2e_1 \otimes f_1 + e_2 \otimes f_0 \\ \alpha(Y^2): &\quad 2e_0 \otimes f_1 + 4e_1 \otimes f_0 \\ \alpha(Y^3): &\quad 6e_0 \otimes f_0 \end{aligned}$$

For ease of reading, let us denote the simple tensors by the letters in the table below at the right.

Now, we know by Prop. 2.2(1) that taking the rational closure gives us primitive forms of the generators above, and thus we obtain the following:

	e_2	e_1	e_0
f_1	a	b	c
f_0	d	e	f

$$W_3^{\mathbb{Q}} = \langle a, 2b+d, c+2e, f\rangle$$

Now we can compute the next component. We have that $\alpha(X)$ sends both of our weight 1 vectors, $e_2 \otimes f_0$ and $e_1 \otimes f_1$, to $e_2 \otimes f_1$. Thus, we compute

$$\ker \alpha(X)|_1 = \ker \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

and so we obtain our next highest weight vector $b-d$. Then, by calculating the Y-action, we obtain $W_1^{\mathbb{Q}} = \langle b-d, e-c\rangle$. If there was another piece to the composition, we would repeat this process.

2.2. Yet another cokernel

There remains a glaring fact to consider: our "decomposition" is not isomorphic to the tensor product, and so it is more a submodule than a decomposition. In particular, though our components are all complemented by Proposition 2.2, their complements are not necessarily always of the form $W_i^{\mathbb{Q}}$. Hence, we can study these complements by forming a new cokernel:

Definition 2.4. Let $n \geq k$, and let V_n, V_k be the previously defined $\mathbb{Z}$-analogues of their respective $\mathfrak{sl}(2,\mathbb{Q})$ irreducible representations. Let $V_n \otimes V_k$ have ZCG $W_{n+k}^{\mathbb{Q}} \oplus \cdots \oplus W_{n-k}^{\mathbb{Q}}$, whose generators $\{g_1,\dots,g_{(n+1)(k+1)}\}$ can be written as a matrix $M = [g_1 \cdots g_{(n+1)(k+1)}]$. Then the *ZCG cokernel* of $V_n \otimes V_k$ is defined as follows:

$$\operatorname{coker} M = \mathbb{Z}^{(n+1)(k+1)}/\langle g_1,\dots,g_{(n+1)(k+1)}\rangle.$$

This can be computed using the Smith normal form; for instance, the ZCG cokernel of $V_2 \otimes V_1$ from Example 2.2 is $(\mathbb{Z}/3)^2$. We have computed this cokernel for small cases ($n, k \leq 9$); they are included in the Appendix.

In the small cases, patterns can already be seen and conjectured based on whether n, k, or their sum is prime. This hints that the properties of this cokernel rely heavily on the primes of $\mathbb{Z}$, an exceedingly complicated structure. Thus, we hope that this cokernel can be further studied through the q-enumerated or quantum group case, in which we compute over $\mathbb{Q}[q,q^{-1}]$.

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Appendix

In the table below, we display computations of ZCG cokernels for small cases of $V_n \otimes V_k$, i.e. for $n, k \leq 9$. These were calculated using SageMath. The program was written following the procedure of Example 2.2. Note that $\mathbb{Z}_n^m$ denotes $(\mathbb{Z}/n)^m$ in the following entries.

Table 1: *ZCG cokernels of small $V_n \otimes V_k$*

$V_n \otimes V_k$	ZCG Cokernel
$V_1 \otimes V_2$	$\mathbb{Z}_3^2$
$V_1 \otimes V_3$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4^2$
$V_1 \otimes V_4$	$\mathbb{Z}_5^4$
$V_1 \otimes V_5$	$\mathbb{Z}_3 \oplus \mathbb{Z}_6^3$
$V_1 \otimes V_6$	$\mathbb{Z}_7^6$
$V_1 \otimes V_7$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4^2 \oplus \mathbb{Z}_8^4$
$V_1 \otimes V_8$	$\mathbb{Z}_3^2 \oplus \mathbb{Z}_9^6$
$V_1 \otimes V_9$	$\mathbb{Z}_5^3 \oplus \mathbb{Z}_{10}^5$
$V_2 \otimes V_3$	$\mathbb{Z}_5^2 \oplus \mathbb{Z}_{30}^2$
$V_2 \otimes V_4$	$\mathbb{Z}_3 \oplus \mathbb{Z}_{30} \oplus \mathbb{Z}_{60}^2$
$V_2 \otimes V_5$	$\mathbb{Z}_7^2 \oplus \mathbb{Z}_{35} \oplus \mathbb{Z}_{60}^2$
$V_2 \otimes V_6$	$\mathbb{Z}_2^4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{28} \oplus \mathbb{Z}_{84}^4$
$V_2 \otimes V_7$	$\mathbb{Z}_3^2 \oplus \mathbb{Z}_{126}^2 \oplus \mathbb{Z}_{252}^4$
$V_2 \otimes V_8$	$\mathbb{Z}_5 \oplus \mathbb{Z}_{30} \oplus \mathbb{Z}_{60}^2 \oplus \mathbb{Z}_{120} \oplus \mathbb{Z}_{360}^3$
$V_2 \otimes V_9$	$\mathbb{Z}_{11}^2 \oplus \mathbb{Z}_{33}^2 \oplus \mathbb{Z}_{495}^6$
$V_3 \otimes V_4$	$\mathbb{Z}_{35}^4 \oplus \mathbb{Z}_{210}^2$
$V_3 \otimes V_5$	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4^3 \oplus \mathbb{Z}_{28} \oplus \mathbb{Z}_{168} \oplus \mathbb{Z}_{840}^3$
$V_3 \otimes V_6$	$\mathbb{Z}_{21}^4 \oplus \mathbb{Z}_{42}^2 \oplus \mathbb{Z}_{420}^4$
$V_3 \otimes V_7$	$\mathbb{Z}_2^6 \oplus \mathbb{Z}_6^2 \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{60}^3 \oplus \mathbb{Z}_{420}^3 \oplus \mathbb{Z}_{840}^2$
$V_3 \otimes V_8$	$\mathbb{Z}_3^4 \oplus \mathbb{Z}_{33}^4 \oplus \mathbb{Z}_{6930}^2 \oplus \mathbb{Z}_{13860}^4$
$V_3 \otimes V_9$	$\mathbb{Z}_2^4 \oplus \mathbb{Z}_{20}^3 \oplus \mathbb{Z}_{220}^2 \oplus \mathbb{Z}_{660}^3 \oplus \mathbb{Z}_{1320} \oplus \mathbb{Z}_{3960}^3$
$V_4 \otimes V_5$	$\mathbb{Z}_{21}^2 \oplus \mathbb{Z}_{42}^2 \oplus \mathbb{Z}_{630}^6$
$V_4 \otimes V_6$	$\mathbb{Z}_2^4 \oplus \mathbb{Z}_{30}^3 \oplus \mathbb{Z}_{420}^4 \oplus \mathbb{Z}_{840}^4$
$V_4 \otimes V_7$	$\mathbb{Z}_3^6 \oplus \mathbb{Z}_{2310}^6 \oplus \mathbb{Z}_{4620}^4$
$V_4 \otimes V_8$	$\mathbb{Z}_3^3 \oplus \mathbb{Z}_6^5 \oplus \mathbb{Z}_{30}^3 \oplus \mathbb{Z}_{660}^4 \oplus \mathbb{Z}_{4620} \oplus \mathbb{Z}_{13860}^2 \oplus \mathbb{Z}_{27720}^2$
$V_4 \otimes V_9$	$\mathbb{Z}_{11}^4 \oplus \mathbb{Z}_{33}^2 \oplus \mathbb{Z}_{429}^4 \oplus \mathbb{Z}_{2145}^2 \oplus \mathbb{Z}_{90090}^2 \oplus \mathbb{Z}_{180180}^4$
$V_5 \otimes V_6$	$\mathbb{Z}_3^4 \oplus \mathbb{Z}_{42}^2 \oplus \mathbb{Z}_{462}^4 \oplus \mathbb{Z}_{6930}^6$
$V_5 \otimes V_7$	$\mathbb{Z}_2^4 \oplus \mathbb{Z}_6^5 \oplus \mathbb{Z}_{12}^4 \oplus \mathbb{Z}_{60}^2 \oplus \mathbb{Z}_{660} \oplus \mathbb{Z}_{9240}^2 \oplus \mathbb{Z}_{27720}^4$
$V_5 \otimes V_8$	$\mathbb{Z}_3^4 \oplus \mathbb{Z}_{33}^6 \oplus \mathbb{Z}_{429}^2 \oplus \mathbb{Z}_{30030}^6 \oplus \mathbb{Z}_{60060}^2 \oplus \mathbb{Z}_{180170}^2$
$V_5 \otimes V_9$	$\mathbb{Z}_2^4 \oplus \mathbb{Z}_6^3 \oplus \mathbb{Z}_{42} \oplus \mathbb{Z}_{462}^4 \oplus \mathbb{Z}_{2310} \oplus \mathbb{Z}_{30030}^2 \oplus \mathbb{Z}_{60060}^5 \oplus \mathbb{Z}_{180180}^2 \oplus \mathbb{Z}_{360360}^2$
$V_6 \otimes V_7$	$\mathbb{Z}_3^2 \oplus \mathbb{Z}_{66}^6 \oplus \mathbb{Z}_{6006}^6 \oplus \mathbb{Z}_{90090}^4 \oplus \mathbb{Z}_{180180}^2$
$V_6 \otimes V_8$	$\mathbb{Z}_2 \oplus \mathbb{Z}_6^3 \oplus \mathbb{Z}_{42}^4 \oplus \mathbb{Z}_{462} \oplus \mathbb{Z}_{924}^4 \oplus \mathbb{Z}_{60060}^5 \oplus \mathbb{Z}_{120120}^2 \oplus \mathbb{Z}_{360360}^4$
$V_6 \otimes V_9$	$\mathbb{Z}_{11}^2 \oplus \mathbb{Z}_{2145}^2 \oplus \mathbb{Z}_{15015}^{10} \oplus \mathbb{Z}_{30030}^6 \oplus \mathbb{Z}_{60060}^2 \oplus \mathbb{Z}_{180180}^2$
$V_7 \otimes V_8$	$\mathbb{Z}_2^6 \oplus \mathbb{Z}_{33}^2 \oplus \mathbb{Z}_{429}^4 \oplus \mathbb{Z}_{4290}^5 \oplus \mathbb{Z}_{30030}^4 \oplus \mathbb{Z}_{90090}^6 \oplus \mathbb{Z}_{180180}^2$
$V_7 \otimes V_9$	$\mathbb{Z}_2^{10} \oplus \mathbb{Z}_4^6 \oplus \mathbb{Z}_{12}^2 \oplus \mathbb{Z}_{132} \oplus \mathbb{Z}_{60060}^2 \oplus \mathbb{Z}_{120120}^{10} \oplus \mathbb{Z}_{240240}^4 \oplus \mathbb{Z}_{720720}^4$
$V_8 \otimes V_9$	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_{78}^2 \oplus \mathbb{Z}_{858}^4 \oplus \mathbb{Z}_{4290}^8 \oplus \mathbb{Z}_{72930}^4 \oplus \mathbb{Z}_{510510}^4 \oplus \mathbb{Z}_{1531530}^6 \oplus \mathbb{Z}_{306306}^2$