# A Sum and Product Game

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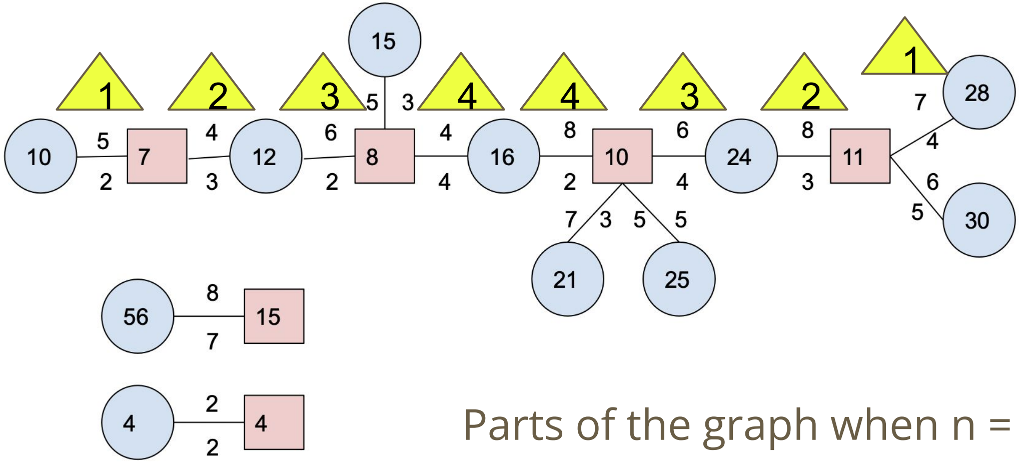
# How does the game work?

- Given two numbers 2 ≤ p, q ≤ n
- The sum person S knows (p+q); the product person P knows (pq)
- They alternatively answer the question: do you know (p, q)?



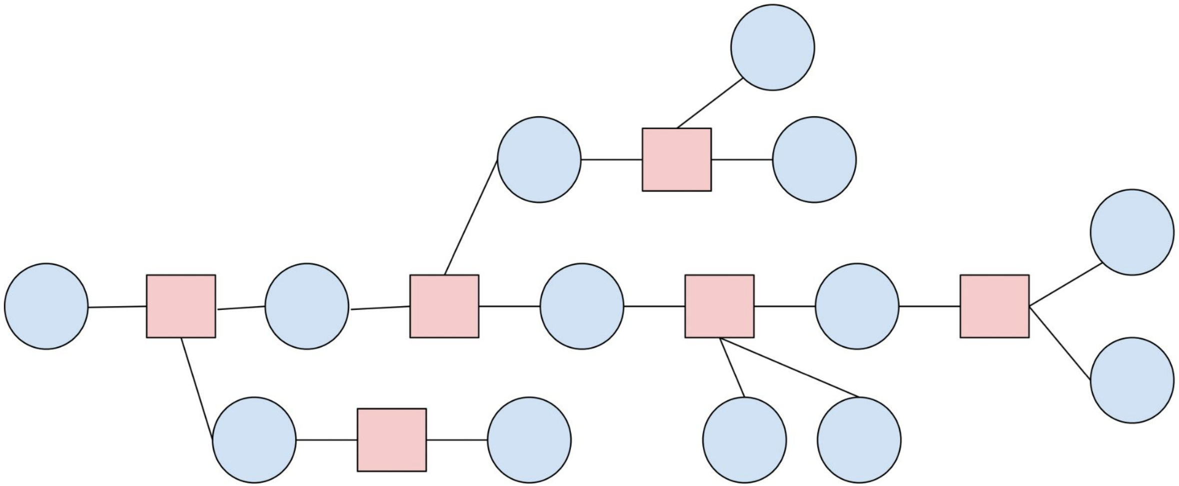

Parts of the graph when n = 8

| | S | P | S | P |
|---|---|---|---|---|
| (5,2) | N | Y | | |
| (4,3) | N | N | Y | |
| (6,2) | N | N | N | Y |

# Relating the Game to the Graph

**Definition 1.** *A sum path $P$ of length $l = l(P)$ in $G(n)$ is two length $l$ sequences $a_i, b_i$ such that $a_i \neq a_{i+1}$, $a_i \geq b_i$ $\forall i$, and $a_{2i}b_{2i} = a_{2i-1}b_{2i-1}$ $\forall 1 \leq i \leq \frac{l}{2}$, $a_{2i} + b_{2i} = a_{2i+1} + b_{2i+1}$ $\forall 1 \leq i \leq \frac{l-1}{2}$.*

**Definition 5.** *A sum tail of length $l$ is a sum path $T = (a_i, b_i)$ of length $l$ such that for every other sum path $\bar{T} = (\bar{a_i}, \bar{b_i})$ of length $\bar{l}$, where $(\bar{a_1}, \bar{b_1}) = (a_1, b_1)$, $\bar{l} \leq l$.*



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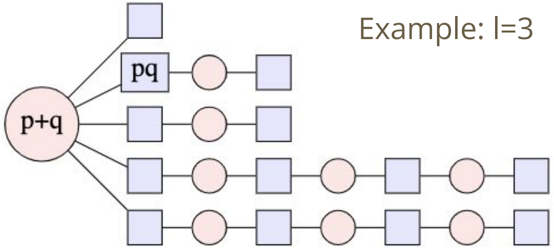
**Theorem.** *A game of $(p, q)$ involves $l = 2m - 1$ NO before YES if and only if*

- *There is one sum tail of length $l$ with $(a_1, b_1) = (p, q)$*
- *There is at least one other sum path of length $l' \geq l$ with $a'_1 + b'_1 = p + q$, $a_1 \neq p$*

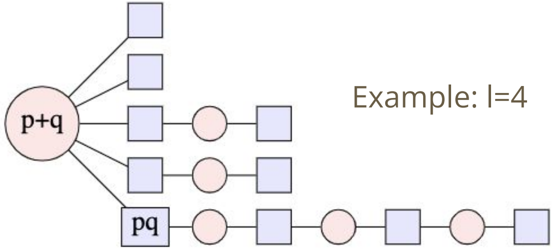
Example: l=3




*A game of $(p, q)$ involves $l = 2m$ NO before YES if and only if*

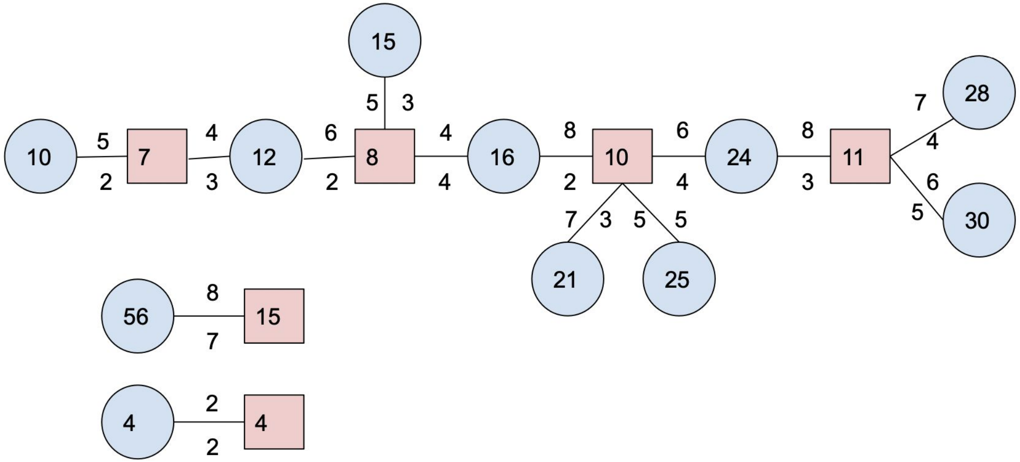
- *There is at least one sum tail of length $l - 1$ with $a_1 + b_1 = p + q$, $a_1 \neq p$*
- *There is one sum path of length $l' > l - 1$ with $(a'_1, b'_1) = (p, q)$*
- *There is no sum path of length $l'' > l - 1$ with $a''_1 + b''_1 = p + q$, $a''_1 \neq p$*

Example: l=4



# Properties of G(n)

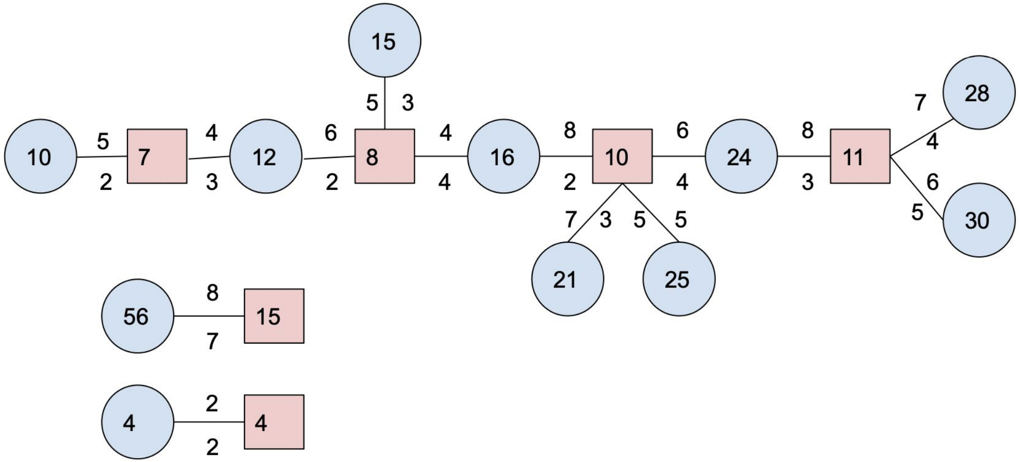
- Each tail either has length one or ends with a product node



# Properties of G(n)

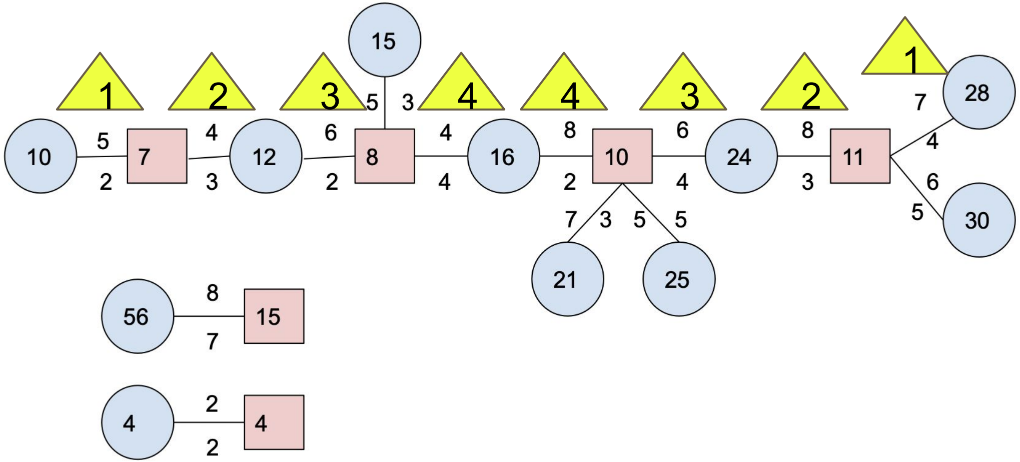
- There is no path from a ‘big’ sum node S to any other sum node

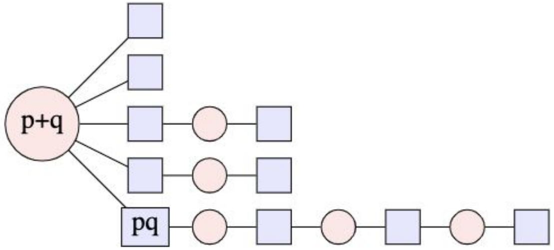
$$3 + 2n - \sqrt{5 + 4n} \leq S \leq 2n$$



# Properties of G(n)

- Suppose for given n, the game of some (p,q) involves r NOs before YES. Then for every r' < r, there is a pair (p', q') such that the game of (p', q') involves r' NOs before YES



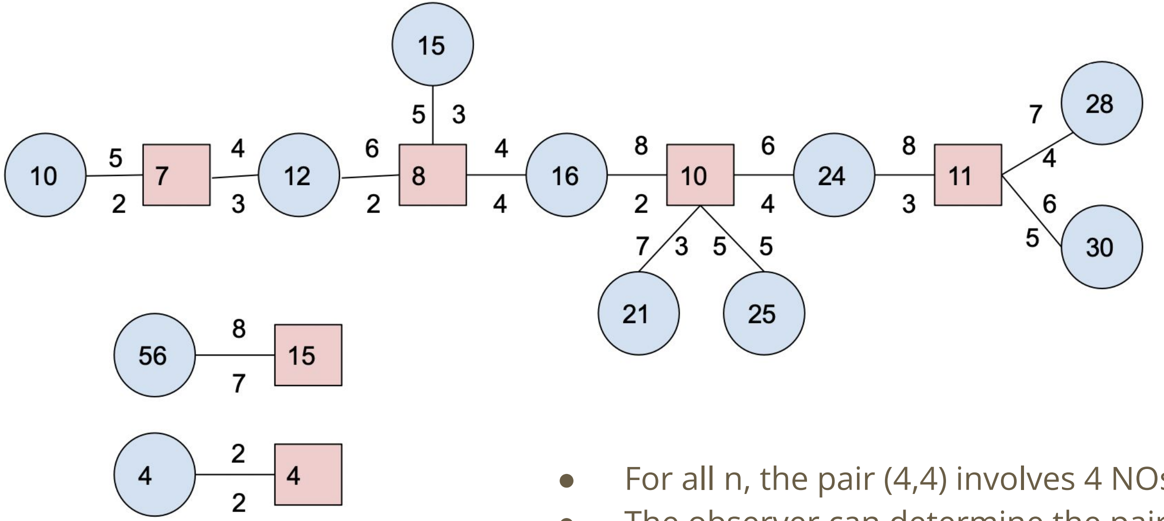
You are the observer of the game and given n ...

# Can you determine the pair of numbers (p,q) after hearing 4 NOs before a YES?

You are the observer of the game …

Are there infinitely many n where you can determine the pair of numbers (p,q) after hearing 4 NOs before a YES?

# Q: Does the observer know (p,q) after 4 NOs and YES?

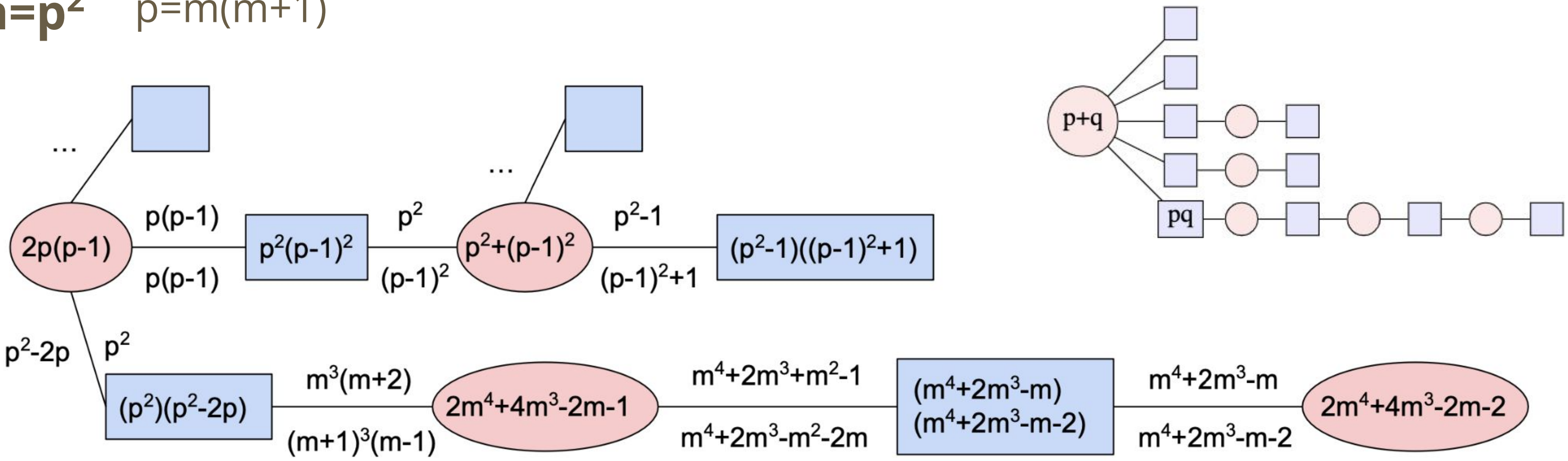



- For all n, the pair (4,4) involves 4 NOs before a YES
- The observer can determine the pair of numbers (p,q) after hearing 4 NOs before a YES if and only if there is no other pair with 4 NOs before a YES.

# ∃ ∞ n s.t. the observer does not know (p,q) after 4 N 1 Y

I.e. ∃ ∞ n such that there is at least another pair (p′, q′) whose game involves 4 NO and YES

**$n=p^2$** $p=m(m+1)$



# ∃ ∞ n s.t. the observer knows (p,q) after 4 N 1 Y

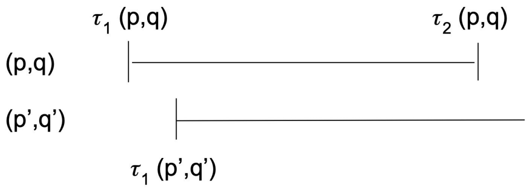
I.e. ∃ ∞ n such that there is no other pair (p', q') whose game involves 4 NO and YES

$$C_n = \{(p,q) \mid (p,q)\, involves\, 4\, NOs\, before\, YES\, in\, game\, of\, n\}$$

**Definition.** *$\tau_1(p,q)$ is the least integer such that $(p,q) \in C_{\tau_1(p,q)}$.*

**Definition.** *$\tau_2(p,q)$ is the greatest integer such that $(p,q) \in C_{\tau_2(p,q)}$.*

**Theorem.** *$\tau_2(C = (p,q))$ is well-defined and $(p,q) = C \in C_\tau$ for every $\tau_1 \le \tau \le \tau_2$.*

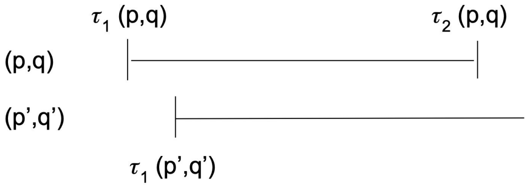



**Goal:** $\mathbb{P}\left( \bigsqcup_{m=\tau_1(C)+1}^{\tau_2(C)+1} [C_{m-1}^c \cap ( \bigcap_{j=m}^{\tau_2(C)+1} C_j)] \neq \emptyset \right) < 1$

# ∃ ∞ n s.t. the observer knows (p,q) after 4 N 1 Y

I.e. ∃ ∞ n such that there is no other pair (p', q') whose game involves 4 NO and YES

**Goal:** $$\mathbb{P}\left(\bigsqcup_{m=\tau_1(C)+1}^{\tau_2(C)+1}\left[C_{m-1}^c \cap \left(\bigcap_{j=m}^{\tau_2(C)+1} C_j\right)\right] \neq \emptyset\right) < 1$$




$$\mathbb{P}\left(\bigsqcup_{m=\tau_1(C)+1}^{\tau_2(C)+1}\left[C_{m-1}^c \cap \left(\bigcap_{j=m}^{\tau_2(C)+1} C_j\right)\right] \neq \emptyset\right) = \sum_{k=1} \mathbb{P}\left(\left\{\bigsqcup_{m=\tau_1(C)+1}^{\tau_2(C)+1}\left[C_{m-1}^c \cap \left(\bigcap_{j=m}^{\tau_2(C)+1} C_j\right)\right] \neq \emptyset\right\} \cap \{(\tau_2(C) - \tau_1(C) = k\}\right)$$

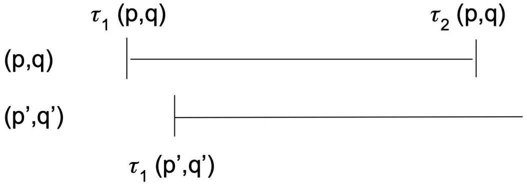
# ∃ ∞ n s.t. the observer knows (p,q) after 4 N 1 Y

I.e. ∃ ∞ n such that there is no other pair (p', q') whose game involves 4 NO and YES

$$|C^c_{j-1}\cap C_j\cap\ldots\cap C_{j+k-1}\cap C^c_{j+k}| = |C^c_{j-1}\cap C_j|[\prod_{l=1}^{k-1}\frac{|C^c_{j-1}\cap C_j\cap\ldots\cap C_{j+l}|}{|C^c_{j-1}\cap C_j\cap\ldots\cap C_{j+l-1}|}]\frac{|C^c_{j-1}\cap C_j\cap\ldots\cap C_{j+k-1}\cap C^c_{j+k}|}{|C^c_{j-1}\cap C_j\cap\ldots\cap C_{j+k-1}|}$$

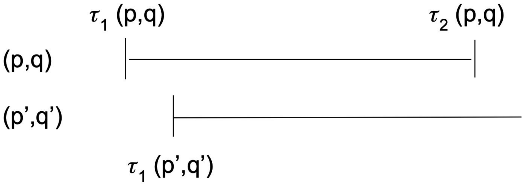
$$\approx |C^c_{j-1}\cap C_j|[\prod_{l=1}^{k-1}\frac{|C_{j+l-1}\cap C_{j+l}|}{|C_{j+l-1}|}](1-\frac{|C_{j+k-1}\cap C_{j+k}|}{|C_{j+k-1}|})$$

$$R=\lim_{j\to\infty}\frac{|C_{j-1}\cap C_j|}{|C_{j-1}|}\qquad \lim_{j\to\infty}|C^c_{j-1}\cap C_j\cap\ldots\cap C_{j+k-1}\cap C^c_{j+k}| = \lim_{j\to\infty}\{|C^c_{j-1}\cap C_j|R^{k-1}(1-R)\}$$



# ∃ ∞ n s.t. the observer knows (p,q) after 4 N 1 Y

I.e. ∃ ∞ n such that there is no other pair (p', q') whose game involves 4 NO and YES




$$\mathbb{P}\Big(\bigsqcup_{m=\tau_1(C)+1}^{\tau_2(C)+1}[C_{m-1}^c \cap (\bigcap_{j=m}^{\tau_2(C)+1} C_j)] \neq \emptyset\Big) \leq \frac{R}{1-R^2}$$

**Goal:** $$\mathbb{P}\Big(\bigsqcup_{m=\tau_1(C)+1}^{\tau_2(C)+1}[C_{m-1}^c \cap (\bigcap_{j=m}^{\tau_2(C)+1} C_j)] \neq \emptyset\Big) < 1 \quad \Rightarrow \quad \frac{R}{1-R^2} < 1$$

# ∃ ∞ n s.t. the observer knows (p,q) after 4 N 1 Y

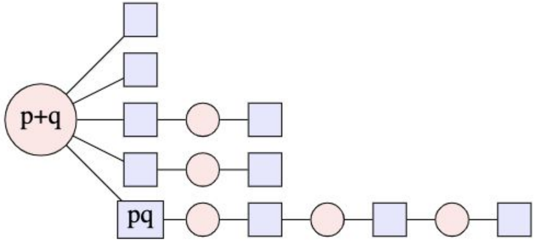
I.e. ∃ ∞ n such that there is no other pair (p', q') whose game involves 4 NO and YES

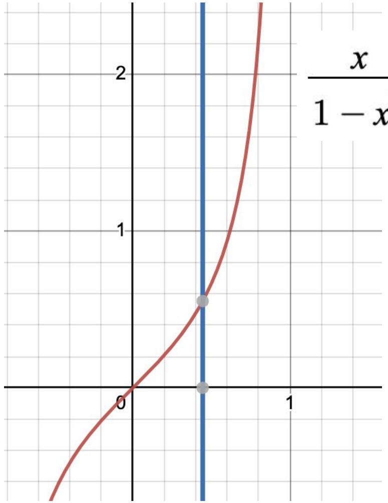
**Goal:** $$\frac{R}{1-R^2} \leq 1$$

$$R = \lim_{j\to\infty} \frac{|C_{j-1} \cap C_j|}{|C_{j-1}|}$$

**1-R=** $$\lim_{j\to\infty} \frac{|C_{j-1} \cap C_j^c|}{|C_{j-1}|} = \lim_{j\to\infty} |S_{2,j} \cup S_{3,j} \cup S_{4,j} \cup S_{5,j}|$$

$$\Rightarrow \quad R \leq \frac{4}{9}$$




$$\frac{x}{1-x^2}$$

$$x = \frac{4}{9}$$

# Thank you!