Code and Geometry	Quantum Error Detection	Detecting One Error	Detecting d Commuting Errors
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Quantum Error Detection and Convex Geometry

Ruochuan Xu

UC Davis REU Greg Kuperberg Group

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Overview

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Quantum Error Detection

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Classical Code and Sphere Packing

Question (Classical Code)

For the Hamming space $H = (\mathbb{Z}/2\mathbb{Z})^n$, find a code $C \subset H$ with maximal dimension that detects errors on d bits.

- Consider the subspace of $(\mathbb{Z}/2\mathbb{Z})^4$ consisting of bit strings of even weight.
- Explicitly, C = {[0000], [1111], [1100], [0011], [1010], [0101], [1001], [0110]}.
- If an error occurs on one bit, the contaminated bit string will no longer lie in C.
- Recall a notion of distance for two bit strings x and y, d(x, y) = weight (x y).
- Bit string in C are spaced apart.
- The minimum distance between two points in C is 2.

Question (Classical Sphere Packing)

Given a metric space (X, d), find the maximal number of disjoint spheres of radius t we can pack into X.

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Question (Classical Sphere Packing)

Given a metric space (X, d), find the maximal number of disjoint spheres of diameter D we can pack into X.

- For a discrete space, it is intuitive and often practical to consider a graph metric
 When the metric is integer-valued, packing spheres of radius t is equivalent to finding a minimum distance set with distance 2t + 1
- For Z² equipped with the "texicab" metric, this is a packing of spheres of radius 1, or equivalently, a minimum distance set of distance 3.



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Detecting d Commuting Errors

Quantum Code and Geometry

Question (Quantum Code)

Given a space of errors \mathcal{E} on a Hilbert space $\mathcal{H} = \mathbb{C}^n$, find a code $\mathcal{C} \subset \mathcal{H}$ with maximal dimension such that \mathcal{C} detects \mathcal{E} .

Question (Convex Geometry)

Given n points in Euclidean space \mathbb{R}^d , find a maximal partition of the n points into r disjoint subsets such that the convex hull spanned by each subset has a common intersection.

■ The convex hulls of $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^d$ consists of points of the form $\sum_{i=1}^m \beta^i \vec{v}_i$, where $\beta^i \in [0, 1]$ and $\sum_i \beta^i = 1$

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Theorem (Quantum Code)

For a space spanned by d commuting errors $\mathcal{E} = \text{span}\{I, E_1, \dots, E_d\}$, on n-dimensional Hilbert space \mathbb{C}^n , there exists a code \mathcal{C} with dimension $\lceil \frac{n}{d+1} \rceil$ that detects \mathcal{E} .

Theorem (Convex Geometry, due to Tverberg)

For any set of n points in d-dimensional Euclidean space \mathbb{R}^d , there exists a partition of the n points into $r = \lceil \frac{n}{d+1} \rceil$ disjoint subsets Y_1, \ldots, Y_r such that $\operatorname{conv}(Y_1) \cap \cdots \cap \operatorname{conv}(Y_r) \neq \emptyset$.

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Theorem

A code $C \subset H$ can detect an error E if the associated projection P_C satisfies

 $P_{\mathcal{C}}EP_{\mathcal{C}} = \epsilon P_{\mathcal{C}}$

for some $\epsilon \in \mathbb{C}$.

- Equivalently, $E|\psi\rangle = \epsilon |\psi\rangle + |\psi^{\perp}\rangle$ for all $|\psi\rangle \in C$, where $|\psi^{\perp}\rangle \perp C$.
- Error detection goes as follows
- Perform a Boolean measurement *i.e.* ask a YES or NO question: Is the state $E | \psi \rangle$ inside C?
- If YES, then the state after measurement is |ψ⟩, and we recovered it uncontaminated.
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Proposition (Suffices to consider a discrete set of errors)

If $\mathcal{C} \subset \mathcal{H}$ can detect both E and F, then it can also detect any linear combination of them.

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- $P_{\mathcal{C}}EP_{\mathcal{C}} = \epsilon(E)P_{\mathcal{C}}$ and $P_{\mathcal{C}}FP_{\mathcal{C}} = \epsilon(F)P_{\mathcal{C}}$
- $\implies P_{\mathcal{C}}(\alpha E + \beta F)P_{\mathcal{C}} = (\alpha \epsilon(E) + \beta \epsilon(F))P_{\mathcal{C}}$

Proposition (Suffices to consider a discrete set of vector state in ${\cal C}$)

The error detection condition can be equivalently formulated using a set of orthonormal basis $\{|\psi_i\rangle\}$ for C:

1.
$$\langle \psi_i | E | \psi_j \rangle = 0$$
 $i \neq j$ 2. $\langle \psi_i | E | \psi_i \rangle = \epsilon(E)$ $\forall i$

• Recall the condition $E|\psi\rangle = \epsilon |\psi\rangle + |\psi^{\perp}\rangle$ for all $|\psi\rangle \in C$.

• Call $\langle \psi_i | E | \psi_i \rangle$ the slope of E w.r.t $| \psi_i \rangle$.

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• Recall the condition $E|\psi\rangle = \epsilon |\psi\rangle + |\psi^{\perp}\rangle$ for all $|\psi\rangle \in C$.

• Call $\langle \psi_i | E | \psi_i \rangle$ the slope of E w.r.t $| \psi_i \rangle$.

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Code and Geometry	Quantum Error Detection	Detecting One Error	Detecting d Commuting Errors
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• Consider $\mathcal{H} = \mathbb{C}^{2n+1}$, $E \in M_n(\mathbb{C})$, $E = E^*$

Label the real eigenvalues of *E* in increasing order as

$$\lambda_{-n} \leq \lambda_{-n+1} \leq \cdots \leq \lambda_0 \leq \ldots \leq \lambda_{n-1} \leq \lambda_n$$

- Denote the eigenstate with eigenvalue λ_k as $|k\rangle$. Consider forming a state as a linear combination of eigenstates.
- If we choose basis elements $\{|\psi_{kl}\rangle\}$ for C each as a linear combination of $|k\rangle$ and $|l\rangle$ for distinct pairs $\{k, l\}$, then we satisfy the 1st condition for error detection

$$\begin{aligned} \langle \psi_{k'l'} | E | \psi_{kl} \rangle &= (\alpha' \langle k' | + \beta' \langle l' |) E(\alpha | k \rangle + \beta | h \rangle) \\ &= (\alpha' \langle k' | + \beta' \langle l' |) (\alpha \lambda_k | k \rangle + \beta \lambda_l | h \rangle) = 0 \quad \{k', l'\} \neq \{k, l\} \end{aligned}$$

Need to satisfy the 2nd condition

$$\langle \psi_{kl} | E | \psi_{kl} \rangle = \epsilon \quad \forall \{k, l\} \text{ for some } \epsilon$$

• For k < l, let $|\psi_{kl}\rangle = \alpha |k\rangle + \beta |l\rangle$, Then

Code and Geometry	Quantum Error Detection	Detecting One Error	Detecting d Commuting Errors
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Code and Geometry	Quantum Error Detection	Detecting One Error	Detecting d Commuting Errors
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$$\begin{aligned} \langle \psi_{kl} | E | \psi_{kl} \rangle &= (\alpha^* \langle k | + \beta^* \langle l | E (\alpha | k \rangle + \beta | l \rangle) \\ &= |\alpha|^2 \lambda_k + |\beta|^2 \lambda_l \\ &= |\alpha|^2 \lambda_k + (1 - |\alpha|^2) \lambda_l \in [\lambda_k, \lambda_l] \\ &= |\alpha|^2 \lambda_k + (1 - |\alpha|^2) \lambda_l \in [\lambda_k, \lambda_l] \end{aligned}$$

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Code and Geometry	Quantum Error Detection	Detecting One Error	Detecting d Commuting Errors
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Code and Geometry	Quantum Error Detection	Detecting One Error	Detecting d Commuting Errors
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Quantum Error Detection and Convex Geometry

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Code and Geometry	Quantum Error Detection	Detecting One Error	Detecting d Commuting Errors
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Quantum Error Detection and Convex Geometry

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Code and Geometry	Quantum Error Detection	Detecting One Error	Detecting d Commuting Errors
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• We have $\langle \psi_{kl} | E | \psi_{kl} \rangle \in [\lambda_k, \lambda_l]$

• Choose $\epsilon = \lambda_0$



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$$C = n + 1$$
 dim $H = 2n + 1$

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Code and Geometry	Quantum Error Detection	Detecting One Error	Detecting d Commuting Errors
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Quantum Error Detection and Convex Geometry

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Quantum Error Detection and Convex Geometry

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- We have $\langle \psi_{kl} | E | \psi_{kl} \rangle \in [\lambda_k, \lambda_l]$
- Choose $\epsilon = \lambda_0$



• The coefficients α_k , β_k can be chosen appropriately such that

 $\langle \psi_k | E | \psi_k \rangle = \lambda_0 \quad \forall k$

 $\dim \mathcal{C} = n+1 \qquad \dim \mathcal{H} = 2n+1$

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Quantum Error Detection and Convex Geometry

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$$\dim \mathcal{C} = n+1 \qquad \dim \mathcal{H} = 2n+1$$

Ruochuan Xu

Code and Geometry	Quantum Error Detection	Detecting One Error	Detecting d Commuting Errors
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- Detecting one error *E* is equivalent to detecting all errors in the error space $\mathcal{E} = \text{span} \{I, E\}$
- Consider $\mathcal{E} = \text{span} \{I, E_1, \dots, E_d\}$, $\mathcal{H} = \mathbb{C}^n$, where $E_a^* = E_a \forall a$ and $E_a E_b = E_b E_a$
- Let $\vec{E} := (E_1, \ldots, E_d)$. Then we can find simultaneous eigenstates $|1\rangle, \ldots, |n\rangle$ such that $\vec{E} |m\rangle = \lambda_m |m\rangle$.
- Consider a subset Y ⊂ {1,..., n}. Form a state as a linear combination of eigenstates with indices in Y:

$$|\psi\rangle \coloneqq \sum_{m \in Y} \sqrt{\beta^m} \, |m\rangle$$

$$\begin{split} \langle \psi | \vec{E} | \psi \rangle &= \Big(\sum_{m' \in Y} \sqrt{\beta^{m'}} \langle m' | \Big) \ \vec{E} \left(\sum_{m \in Y} \sqrt{\beta^m} | m \rangle \right) \\ &= \Big(\sum_{m' \in Y} \sqrt{\beta^{m'}} \langle m | \Big) \left(\sum_{m \in Y} \vec{\lambda}_m \sqrt{\beta^m} | m \rangle \right) \quad = \sum_{m \in Y} \beta^m \vec{\lambda}_m \end{split}$$

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Quantum Error Detection and Convex Geometry

Code and Geometry	Quantum Error Detection	Detecting One Error	Detecting d Commuting Errors
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- Detecting one error E is equivalent to detecting all errors in the error space
 \$\mathcal{E}\$ = span {I, E}
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Ruochuan Xu Quantum Error Detection and Convex Geometry

Code and Geometry	Quantum Error Detection	Detecting One Error	Detecting d Commuting Errors
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Ruochuan Xu Quantum Error Detection and Convex Geometry

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Code and Geometry	Quantum Error Detection	Detecting One Error	Detecting d Commuting Errors
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Ruochuan Xu Quantum Error Detection and Convex Geometry

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Code and Geometry	Quantum Error Detection	Detecting One Error	Detecting d Commuting Errors
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Code and Geometry G	Quantum Error Detection	Detecting One Error	Detecting d Commuting Errors
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- Consider a partition of $\{1, \ldots, n\}$ into *r* disjoint subsets $\{Y_k\}$.
- Choose basis elements of C each as a linear combination of eigenstates with indices in Y_k :

$$|\psi_k\rangle = \sum_{m \in Y_k} \sqrt{\beta_k^m} |m\rangle$$

- Already satisfies the 1st condition for error detection: $\langle \psi_k | \vec{E} | \psi_l \rangle = 0$ $k \neq l$
- Need to choose $\epsilon_1, \ldots, \epsilon_d$ and coefficients β_k^m such that $\langle \psi_k | E_a | \psi_k \rangle = \epsilon_a$ for $a = 1, \ldots, d$ and for all k.
- Equivalently, need to find $\vec{\epsilon} \in \mathbb{R}^d$ and β_k^m such that $\langle \psi_k | \vec{E} | \psi_k \rangle = \vec{\epsilon} \quad \forall k$
- As previously calculated,

$$\langle \psi_k | \vec{E} | \psi_k \rangle = \sum_{m \in Y_k} \beta_k^m \vec{\lambda}_m,$$

where the coefficients β_k^m satisfy $\sum_{m \in Y_k} \beta_k^m = 1$.

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Code and Geometry	Quantum Error Detection	Detecting One Error	Detecting d Commuting Errors
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- Already satisfies the 1st condition for error detection: $\langle \psi_k | \vec{E} | \psi_l \rangle = 0$ $k \neq l$
- Need to choose $\epsilon_1, \ldots, \epsilon_d$ and coefficients β_k^m such that $\langle \psi_k | E_a | \psi_k \rangle = \epsilon_a$ for $a = 1, \ldots, d$ and for all k.
- Equivalently, need to find $\vec{\epsilon} \in \mathbb{R}^d$ and β_k^m such that $\langle \psi_k | \vec{E} | \psi_k \rangle = \vec{\epsilon} \quad \forall k$
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$$\langle \psi_k | \vec{E} | \psi_k \rangle = \sum_{m \in Y_k} \beta_k^m \vec{\lambda}_m \in \operatorname{conv} \left(\{ \vec{\lambda}_i \}_{i \in Y_k} \right).$$

• Therefore, $\vec{\epsilon}$ and β_k^m satisfying the 2nd condition exist if and only if

 $\iff \bigcap \operatorname{conv}(\{\vec{\lambda}_m\}_{m \in Y_k}) \neq \emptyset \quad (\star)$

Then $\vec{\epsilon}$ can be chosen to be any point in the interesection of the convex hulls.

By Tverberg's theorem, there exists a partition of $\{1, ..., n\}$ into $r = \lceil \frac{n}{d+1} \rceil$ disjoint subsets Y_k such that (*) holds

• dim $\mathcal{C} \geq \left\lceil \frac{n}{d+1} \right\rceil$.

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Key Points

- maximize dim $C \implies$ maximize the size of partition of points such that the convex hull spanned by each subset has a common intersection.
- The continuous problem of quantum error detection is discretized and geometrized.

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Thank you

Ruochuan Xu Quantum Error Detection and Convex Geometry University of Chicago

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