Quantum Error Detection and Convex Geometry

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Overview

1. Code and Geometry
2. Quantum Error Detection
3. Detecting One Error
4. Detecting d Commuting Errors

Classical Code and Sphere Packing

Question (Classical Code)

For the Hamming space $H = (\mathbb{Z}/2\mathbb{Z})^n$, find a code $C \subset H$ with maximal dimension that detects errors on d bits.

- Consider the subspace of $(\mathbb{Z}/2\mathbb{Z})^4$ consisting of bit strings of even **weight**.
- Explicitly, $C = \{[0000], [1111], [1100], [0011], [1010], [0101], [1001], [0110]\}$.
- If an error occurs on one bit, the contaminated bit string will no longer lie in C.
- Recall a notion of distance for two bit strings x and y, $d(x, y) = \text{weight}\ (x - y)$.
- Bit string in C are spaced apart.
- The **minimum distance** between two points in C is 2.

Question (Classical Sphere Packing)

Given a metric space (X, d), find the maximal number of disjoint spheres of radius t we can pack into X.

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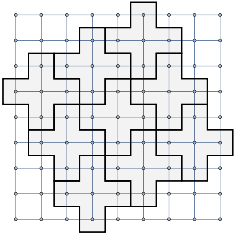
Given a metric space (X, d), find the maximal number of disjoint spheres of diameter D we can pack into X.

- For a discrete space, it is intuitive and often practical to consider a **graph metric**
- When the metric is integer-valued, packing spheres of radius t is equivalent to finding a **minimum distance set** with distance $2t + 1$
- For $\mathbb{Z}^2$ equipped with the "texicab" metric, this is a packing of spheres of radius 1, or equivalently, a minimum distance set of distance 3.



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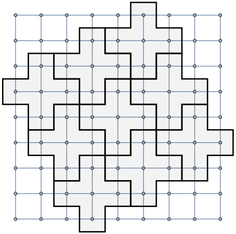

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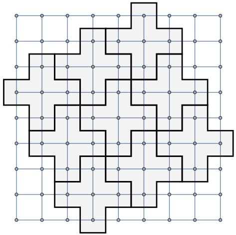

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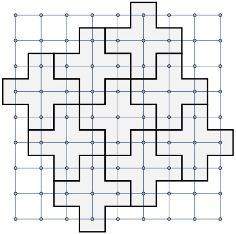

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Question (Quantum Code)

Given a space of errors $\mathcal{E}$ on a Hilbert space $\mathcal{H} = \mathbb{C}^n$, find a code $\mathcal{C} \subset \mathcal{H}$ with maximal dimension such that $\mathcal{C}$ detects $\mathcal{E}$.

Question (Convex Geometry)

Given n points in Euclidean space $\mathbb{R}^d$, find a maximal partition of the n points into r disjoint subsets such that the convex hull spanned by each subset has a common intersection.

- The convex hulls of $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^d$ consists of points of the form $\sum_{i=1}^m \beta^i \vec{v}_i$, where $\beta^i \in [0,1]$ and $\sum_i \beta^i = 1$

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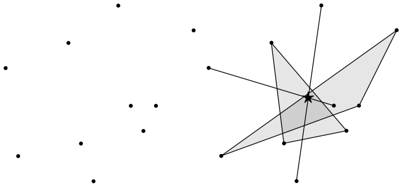

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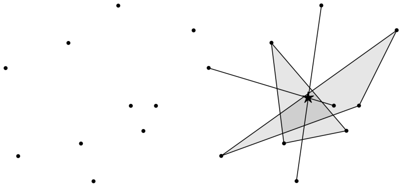

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Theorem (Quantum Code)

For a space spanned by d commuting errors $\mathcal{E} = span\{I, E_1, \ldots, E_d\}$, on n-dimensional Hilbert space $\mathbb{C}^n$, there exists a code $\mathcal{C}$ with dimension $\lceil \frac{n}{d+1} \rceil$ that detects $\mathcal{E}$.

Theorem (Convex Geometry, due to Tverberg)

For any set of n points in d-dimensional Euclidean space $\mathbb{R}^d$, there exists a partition of the n points into $r = \lceil \frac{n}{d+1} \rceil$ disjoint subsets $Y_1, \ldots, Y_r$ such that $conv(Y_1) \cap \cdots \cap conv(Y_r) \neq \emptyset$.

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Quantum Error Detection Condition

Theorem

A code $\mathcal{C} \subset \mathcal{H}$ can detect an error E if the associated projection P_C satisfies

$$P_\mathcal{C} E P_\mathcal{C} = \epsilon P_\mathcal{C}$$

for some $\epsilon \in \mathbb{C}$.

- Equivalently, $E|\psi\rangle = \epsilon|\psi\rangle + |\psi^\perp\rangle$ for all $|\psi\rangle \in \mathcal{C}$, where $|\psi^\perp\rangle \perp \mathcal{C}$.
- Error detection goes as follows
- Perform a Boolean measurement *i.e.* ask a YES or NO question: Is the state $E|\psi\rangle$ inside $\mathcal{C}$?
- If YES, then the state after measurement is $|\psi\rangle$, and we recovered it uncontaminated.
- If NO, then we detect an error, and the state after measurement lies in $\mathcal{C}^\perp$.

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Discretization of Error

Proposition (Suffices to consider a discrete set of errors)

If $\mathcal{C} \subset \mathcal{H}$ can detect both E and F, then it can also detect any linear combination of them.

- $P_C E P_C = \epsilon(E) P_C$ and $P_C F P_C = \epsilon(F) P_C$
- $\implies P_C(\alpha E + \beta F) P_C = (\alpha\epsilon(E) + \beta\epsilon(F)) P_C$

Proposition (Suffices to consider a discrete set of vector state in $\mathcal{C}$)

The error detection condition can be eqiuvalently formulated using a set of orthonormal basis $\{|\psi_i\rangle\}$ for $\mathcal{C}$:

$$1.\ \langle\psi_i|E|\psi_j\rangle = 0 \quad i \neq j \qquad 2.\ \langle\psi_i|E|\psi_i\rangle = \epsilon(E) \quad \forall i$$

- Recall the condition $E|\psi\rangle = \epsilon|\psi\rangle + |\psi^\perp\rangle$ for all $|\psi\rangle \in \mathcal{C}$.
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- $P_{\mathcal{C}} E P_{\mathcal{C}} = \epsilon(E) P_{\mathcal{C}}$ and $P_{\mathcal{C}} F P_{\mathcal{C}} = \epsilon(F) P_{\mathcal{C}}$
- $\implies P_{\mathcal{C}}(\alpha E + \beta F) P_{\mathcal{C}} = (\alpha \epsilon(E) + \beta \epsilon(F)) P_{\mathcal{C}}$

Proposition (Suffices to consider a discrete set of vector state in $\mathcal{C}$)

The error detection condition can be eqiuvalently formulated using a set of orthonormal basis $\{|\psi_i\rangle\}$ for $\mathcal{C}$:

$$1.\ \langle \psi_i | E | \psi_j \rangle = 0 \quad i \neq j \qquad 2.\ \langle \psi_i | E | \psi_i \rangle = \epsilon(E) \quad \forall i$$

- Recall the condition $E|\psi\rangle = \epsilon|\psi\rangle + |\psi^{\perp}\rangle$ for all $|\psi\rangle \in \mathcal{C}$.
- Call $\langle \psi_i | E | \psi_i \rangle$ the **slope** of E w.r.t $|\psi_i\rangle$.

Detecting One Error

- Consider $\mathcal{H} = \mathbb{C}^{2n+1}$, $E \in M_n(\mathbb{C})$, $E = E^*$
- Label the real eigenvalues of E in increasing order as

$$\lambda_{-n} \leq \lambda_{-n+1} \leq \cdots \leq \lambda_0 \leq \ldots \leq \lambda_{n-1} \leq \lambda_n$$

- Denote the eigenstate with eigenvalue λ_k as $|k\rangle$. Consider forming a state as a linear combination of eigenstates.
- If we choose basis elements $\{|\psi_{kl}\rangle\}$ for $\mathcal{C}$ each as a linear combination of $|k\rangle$ and $|l\rangle$ for distinct pairs $\{k, l\}$, then we satisfy the 1st condition for error detection

$$\begin{aligned}\langle\psi_{k'l'}|E|\psi_{kl}\rangle &= (\alpha' \langle k'| + \beta' \langle l'|)E(\alpha |k\rangle + \beta |l\rangle) \\ &= (\alpha' \langle k'| + \beta' \langle l'|)(\alpha\lambda_k |k\rangle + \beta\lambda_l |l\rangle) = 0 \quad \{k', l'\} \neq \{k, l\}\end{aligned}$$

- Need to satisfy the 2nd condition

$$\langle\psi_{kl}|E|\psi_{kl}\rangle = \epsilon \quad \forall\{k, l\} \quad \text{for some } \epsilon$$

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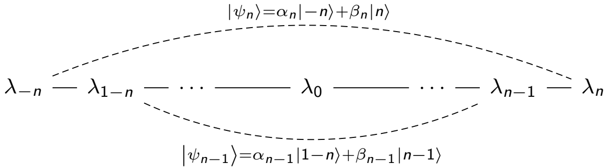

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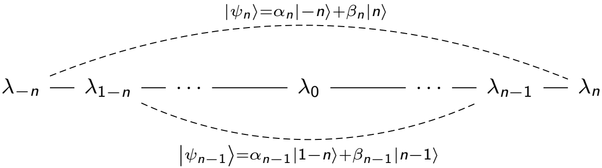

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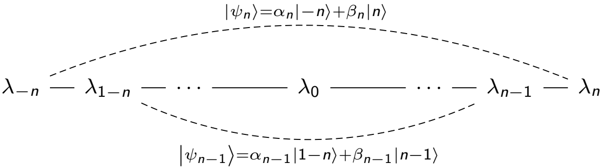

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Detecting d Commuting Errors

- Detecting one error E is equivalent to detecting all errors in the error space $\mathcal{E} = \text{span}\{I, E\}$
- Consider $\mathcal{E} = \text{span}\{I, E_1, \ldots, E_d\}$, $\mathcal{H} = \mathbb{C}^n$, where $E_a^* = E_a \, \forall a$ and $E_a E_b = E_b E_a$
- Let $\vec{E} := (E_1, \ldots, E_d)$. Then we can find simultaneous eigenstates $|1\rangle, \ldots, |n\rangle$ such that $\vec{E}|m\rangle = \vec{\lambda}_m |m\rangle$.
- Consider a subset $Y \subset \{1, \ldots, n\}$. Form a state as a linear combination of eigenstates with indices in Y:

$$|\psi\rangle := \sum_{m \in Y} \sqrt{\beta^m} |m\rangle$$

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- Already satisfies the 1st condition for error detection: $\langle \psi_k | \vec{E} | \psi_l \rangle = 0 \quad k \neq l$
- Need to choose $\epsilon_1, \ldots, \epsilon_d$ and coefficients β_k^m such that $\langle \psi_k | E_a | \psi_k \rangle = \epsilon_a$ for $a = 1, \ldots, d$ and for all k.
- Equivalently, need to find $\vec{\epsilon} \in \mathbb{R}^d$ and β_k^m such that $\langle \psi_k | \vec{E} | \psi_k \rangle = \vec{\epsilon} \quad \forall k$
- As previously calculated,

$$\langle \psi_k | \vec{E} | \psi_k \rangle = \sum_{m \in Y_k} \beta_k^m \vec{\lambda}_m,$$

where the coefficients β_k^m satisfy $\sum_{m \in Y_k} \beta_k^m = 1$.

Detecting d Commuting Errors

- $\langle \psi_k | \vec{E} | \psi_k \rangle = \sum_{m \in Y_k} \beta_k^m \vec{\lambda}_m \in \text{conv}\left(\{\vec{\lambda}_i\}_{i \in Y_k}\right)$.
- Therefore, $\vec{\epsilon}$ and β_k^m satisfying the 2nd condition exist if and only if

$$\iff \bigcap \text{conv}\left(\{\vec{\lambda}_m\}_{m \in Y_k}\right) \neq \emptyset \quad (\star)$$

- Then $\vec{\epsilon}$ can be chosen to be any point in the interesection of the convex hulls.
- By Tverberg's theorem, there exists a partition of $\{1, \ldots, n\}$ into $r = \lceil \frac{n}{d+1} \rceil$ disjoint subsets Y_k such that $(\star)$ holds
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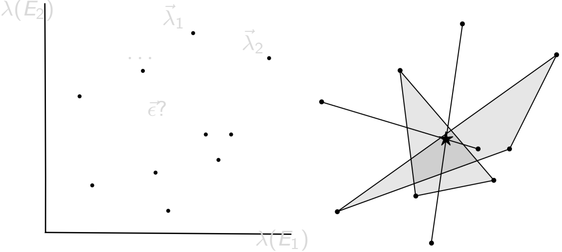

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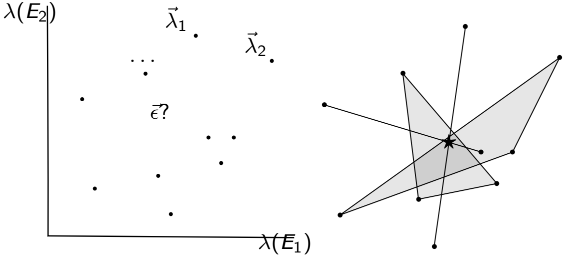

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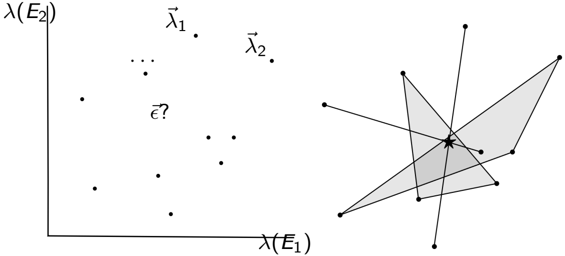

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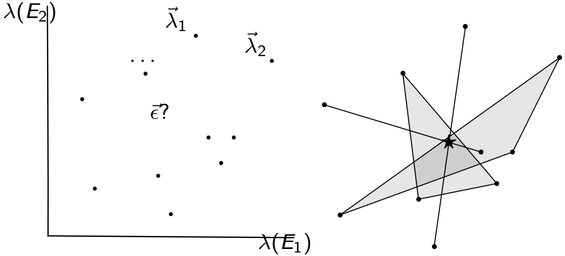

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Thank you