Lie Algebras, Quantum Metrics, and Error Detecting Codes

UC Davis Math REU 2022

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Table of Contents

Matrix Lie Algebras

In general an *algebra* is a set endowed with two operations thought of as addition and multiplication.

Definition

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Definition

Suppose $\mathfrak{g}$ is a vector subspace of $M_n(\mathbb{R})$ or $M_n(\mathbb{C})$ that is closed under the Lie bracket. The set $\mathfrak{g}$, endowed with the operations of $+$ and $[\cdot, \cdot]$, is said to be a *matrix Lie algebra*.

Note: There exists an abstract definition of a Lie algebra, but every (finite-dimensional) abstract Lie algebra is isomorphic to a matrix Lie algebra.

Matrix Lie Algebra Examples

- $\mathfrak{gl}(n, \mathbb{C}) = \{\text{all } n \times n \text{ complex matrices}\}$
- $\mathfrak{sl}(n, \mathbb{C}) = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid \mathrm{tr}(X) = 0\}$
- $\mathfrak{su}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid \mathrm{tr}(X) = 0 \text{ and } X^* = -X\}$

We are most interested in $\mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{su}(n)$ for small values of n.

Irreducible Representations

A *representation* of a Lie algebra $\mathfrak{g}$ is a vector space V together with a linear map $\rho : \mathfrak{g} \to \mathcal{L}(V)$. We can think of each $X \in \mathfrak{g}$ as being a linear operator on V, so we say $\mathfrak{g}$ *acts on* V.

This map must preserve the Lie algebra structure of $\mathfrak{g}$, so we require

$$\rho([X, Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X)$$

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Example

For a matrix Lie algebra $\mathfrak{g} \subseteq M_n(\mathbb{C})$, we can let $V = \mathbb{C}^n$ and $\rho(X) = X$. This is called the *defining representation* of $\mathfrak{g}$.

A representation is said to be *irreducible* if it has no non-trivial suprepresentations – i.e. subspaces W of V such that ρ restricted to W is a representation.

Some irreducible representations of $\mathfrak{sl}(d, \mathbb{C})$

Let $\mathcal{H}_k = \mathbb{C}[x_1, \ldots, x_d]_k$, the vector space of homogeneous polynomials in $x_1, \ldots, x_d$ of degree k.

Example

Taking $\mathfrak{sl}(3, \mathbb{C})$, we have

$$\mathcal{H}_1 = \operatorname{span}_{\mathbb{C}}\{x, y, z\}$$
$$\mathcal{H}_2 = \operatorname{span}_{\mathbb{C}}\{x^2, y^2, z^2, xy, xz, yz\}$$
$$\mathcal{H}_3 = \operatorname{span}_{\mathbb{C}}\{x^3, y^3, z^3, x^2y, xy^2, x^2z, xz^2, y^2z, yz^2, xyz\}$$

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$$\begin{aligned} \mathcal{H}_1 &= \mathrm{span}_{\mathbb{C}}\{x, y, z\} \\ \mathcal{H}_2 &= \mathrm{span}_{\mathbb{C}}\{x^2, y^2, z^2, xy, xz, yz\} \\ \mathcal{H}_3 &= \mathrm{span}_{\mathbb{C}}\{x^3, y^3, z^3, x^2y, xy^2, x^2z, xz^2, y^2z, yz^2, xyz\} \end{aligned}$$

$\mathcal{H}_k$ can be made into an irreducible representation of $\mathfrak{sl}(d, \mathbb{C})$ under the identification

$$\rho(E_{ij}) = x_j \frac{\partial}{\partial x_i}$$

This representation is isomorphic to the k^{th} symmetric power of the defining representation.

Example: $\mathcal{H}_3$ for $\mathfrak{sl}(3, \mathbb{C})$

The representation $\mathcal{H}_3$ of $\mathfrak{sl}(3, \mathbb{C})$ may be summarized in the following diagram.

$$\rho(E_{11}) = x\frac{\partial}{\partial x}$$

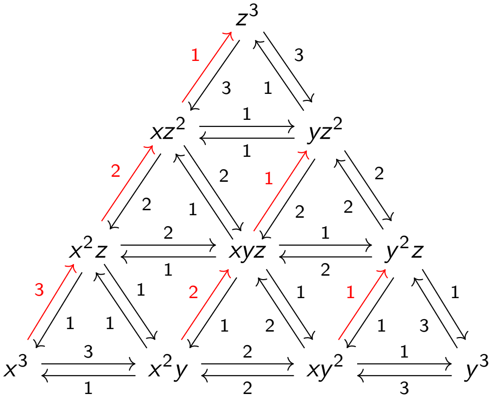
$$\rho(E_{12}) = y\frac{\partial}{\partial x}$$

$$\rho(E_{13}) = z\frac{\partial}{\partial x}$$

$$\rho(E_{21}) = x\frac{\partial}{\partial y}$$

$$\rho(E_{22}) = y\frac{\partial}{\partial y}$$

$$\vdots$$



Example: $\mathcal{H}_3$ for $\mathfrak{sl}(3, \mathbb{C})$

Rewriting $\mathcal{H}_3$ with basis vectors $|abc\rangle = \sqrt{\binom{k}{a,b,c}}\, x^a y^b z^c$, the diagram simplifies considerably.

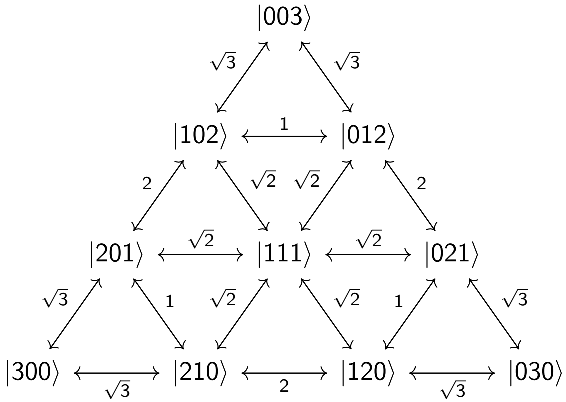


Table of Contents

Quantum Metric Spaces

Suppose $\mathcal{H} = \mathbb{C}^d$ is the state space of a quantum system. Let $\mathcal{L}(\mathcal{H}) = M_d(\mathbb{C})$ denote the set of linear operators from $\mathcal{H}$ to itself. Elements of $\mathcal{L}(\mathcal{H})$ are interpreted as errors on $\mathcal{H}$.

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A quantum metric assigns a real number to each error representing its severity. In particular, a quantum metric may be defined in terms of a function $D : M_d(\mathbb{C}) \to [0, \infty]$ satisfying

- $D(XY) \leq D(X) + D(Y)$
- $D(X + Y) \leq \max\{D(X), D(Y)\}$
- $D(X^*) = D(X)$
- $D(\alpha X) = D(X)$ for $\alpha \neq 0$
- $D(X) = 0$ if and only if $X = \alpha I$ for some $\alpha \in \mathbb{C}$

In error correction problems, we often assume that more severe errors are much less likely to occur.

Quantum Metric Spaces, Continued

Given such a function D, for each $t \in [0, \infty]$ we may define

$$\mathcal{V}_t = \{X \in M_d(\mathbb{C}) : D(X) \leq t\}$$

The collection $\{\mathcal{V}_t\}$ is called a $*$-algebra filtration of $M_d(\mathbb{C})$. A quantum metric may be equivalently defined in terms of this filtration.

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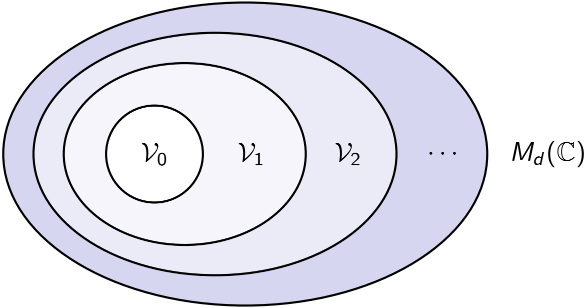

Quantum Metric Spaces of Lie Type

Example

Let $\mathcal{E}$ be any subspace of $M_d(\mathbb{C})$ such that $I \in \mathcal{E}$ and $\mathcal{E}^* = \mathcal{E}$. We can build a quantum metric as follows:

$$\mathcal{V}_0 = \operatorname{span}_{\mathbb{C}}\{I\}, \qquad \mathcal{V}_1 = \mathcal{E}, \qquad \mathcal{V}_n = \operatorname{span}_{\mathbb{C}} \mathcal{E}^n \text{ for } n = 2, 3, 4, \ldots$$

Quantum metrics constructed in this way are called *quantum graph metrics*.

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Quantum metrics constructed in this way are called *quantum graph metrics*.

Suppose $\mathcal{H} \cong \mathbb{C}^n$ is a representation of $\mathfrak{g}$ with representation map $\rho : \mathfrak{g} \to M_n(\mathbb{C})$. If we construct a quantum graph metric with $\mathcal{E} = \operatorname{span}_{\mathbb{C}}\{I\} \oplus \rho(\mathfrak{g})$, then the resulting quantum metric space has many nice properties. We say quantum metric spaces of this form are of *Lie type*.

$\mathfrak{sl}(3,\mathbb{C})$ Quantum Metric Spaces

Recall the diagram for the representation $\mathcal{H}_3$ of $\mathfrak{sl}(3,\mathbb{C})$:

In the corresponding quantum metric, distance one errors take vectors to adjacent ones in the diagram.

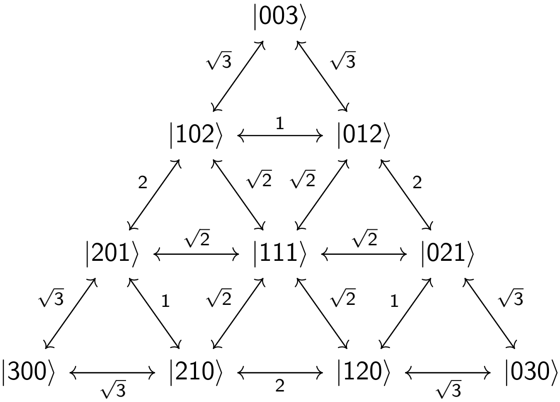

Table of Contents

Quantum Error Detecting Codes

A *quantum code* is a subspace $\mathcal{C}$ of $\mathcal{H}$.

Suppose we wish to send the message $|\psi\rangle \in \mathcal{C}$ and an error E occurs, meaning $E|\psi\rangle$ is received. This is fine if

- $E|\psi\rangle = \varepsilon|\psi\rangle$ (in which case the error E is *inconsequential*), or
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An operator equation that encapsulates both of these scenarios is

$$P_{\mathcal{C}} E P_{\mathcal{C}} = \varepsilon P_{\mathcal{C}},$$

where $P_{\mathcal{C}}$ denotes the orthogonal projection onto $\mathcal{C}$. Hence, we say a code $\mathcal{C}$ can detect errors of distance t if $P_{\mathcal{C}} E P_{\mathcal{C}} = \varepsilon(E) P_{\mathcal{C}}$ for all E in $\mathcal{V}_t$. This ε is called the *slope* of the code.

KLV codes

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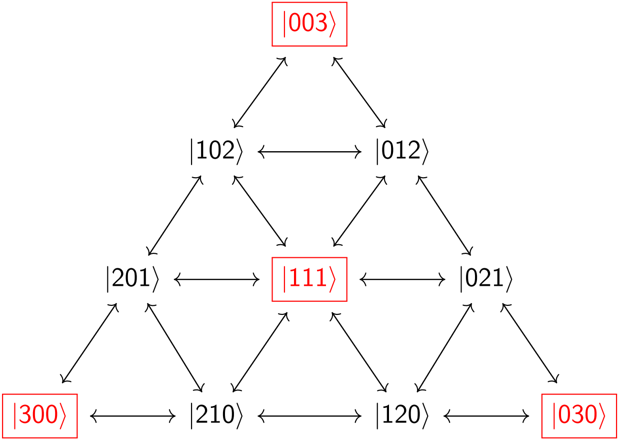
Suppose we wish to detect errors from $\mathcal{V}_t$ for some t.

1. Find a subspace $\mathcal{B}$ of $\mathcal{H}$ such that $\mathcal{V}_t$ restricted to $\mathcal{B}$ is commutative.
2. Find a subspace $\mathcal{C}$ of $\mathcal{B}$ that detects those commutative errors. This reduces to a convex geometry problem.

In their original paper, KLV used a greedy algorithm for step 1, and cited Tverberg's theorem for step 2. With knowledge of the structure of the $\mathfrak{sl}(3, \mathbb{C})$ quantum metric spaces, both of these can be improved!

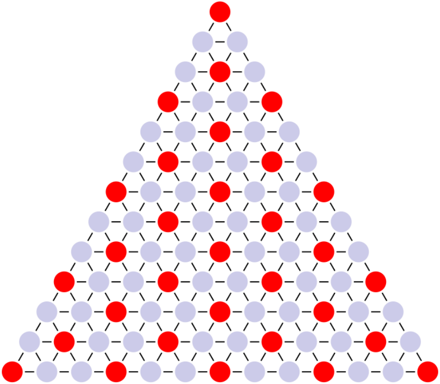
Finding a commutative subspace



If we choose a subspace spanned by vectors spaced out by distance $t+1$, then the only non-zero surviving errors of distance $\leq t$ will be diagonal.

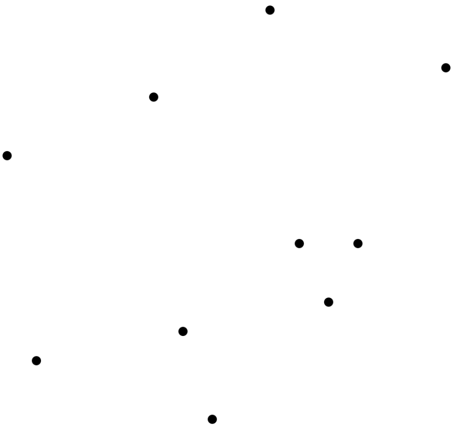
A larger example

$\mathcal{H}_{12}$ is shown below. Looking for subspaces $\mathcal{B}$ that diagonalize $\mathcal{V}_1$, we can achieve $\dim \mathcal{B} = \left\lceil \frac{\dim \mathcal{H}_k}{3} \right\rceil$. The greedy algorithm given by KLV gives $\dim \mathcal{B} \geq \left\lceil \frac{\dim \mathcal{H}_k}{8} \right\rceil$.



The Tverberg problem

Given n points in $\mathbb{R}^d$, we wish to partition them into subsets so that the intersection of the convex hull of each subset is nonempty.



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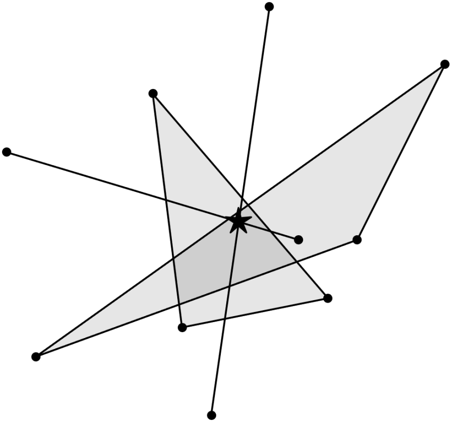

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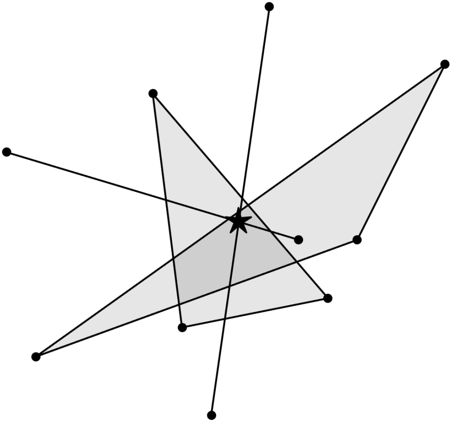

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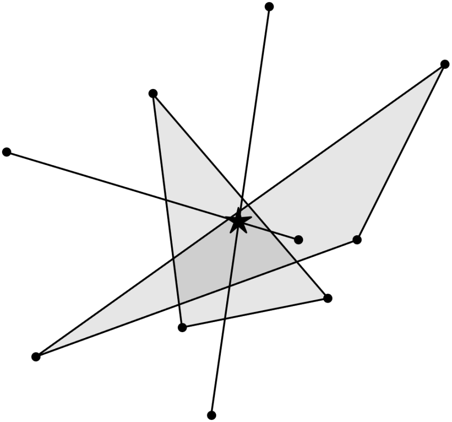

What is the maximal number of subsets we can take?

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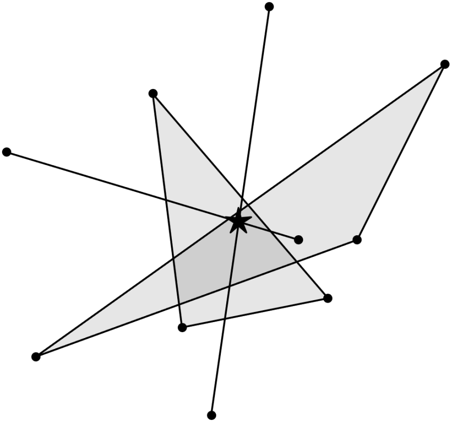

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However, for highly ordered sets of points, we can potentially do better!

KLV Construction, Step 2

Suppose we have a set of commuting errors $\mathcal{F}$ with basis $\{F_1, \ldots, F_d\}$ on $\mathcal{B}$. Since they commute, there is a basis in which all are diagonal. To each basis vector $|m\rangle$, we can associate a vector $\vec{\lambda}_m = (\lambda_m^{(1)}, \ldots, \lambda_m^{(d)}) \in \mathbb{R}^d$, where $\lambda_m^{(j)}$ is the eigenvalue of $|m\rangle$ for the matrix F_j.

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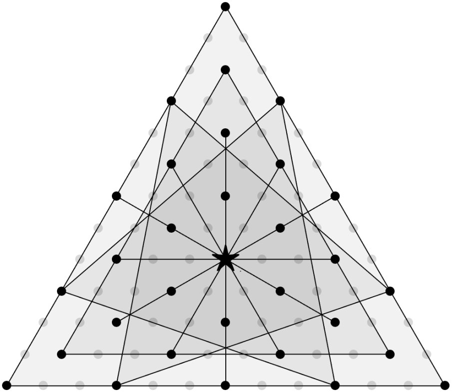
To find a code $\mathcal{C}$ inside $\mathcal{B}$, we find a Tverberg partition of the λ_i's. The Tverberg point, $\vec{\varepsilon}$ will become the slope of the code, and for each set in the partition, we can construct a corresponding vector $|\psi\rangle$ in $\mathcal{C}$ with $\langle\psi| F_j |\psi\rangle = \varepsilon_j$. Hence, the number of sets in the partition is the dimension of $\mathcal{C}$.

Super Tverberg points



For our earlier example of an subspace $\mathcal{B}$ of $\mathcal{H}_k$ for $\mathfrak{sl}(3, \mathbb{C})$, the collection of points is a triangular lattice. By pairing up points on opposite sides of the centroid, we can get approximately $4n/9$ sets. Hence,

$$\frac{\dim \mathcal{C}}{\dim \mathcal{B}} = \frac{4}{9} + O(1/k)$$

which implies

$$\frac{\dim \mathcal{C}}{\dim \mathcal{H}_k} = \frac{4}{27} + O(1/k).$$

Further questions

- The KLV construction can sometimes be modified to work with *block* diagonal error, not just diagonal error. This allows us to enlarge $\mathcal{B}$. How much advantage does this give?
- What about $\mathfrak{sl}(4)$ and beyond?

Thank you!