q-Carlitz Matrices and their conjectured Smith Normal Forms over $\mathbb{Z}[q, q^{-1}]$

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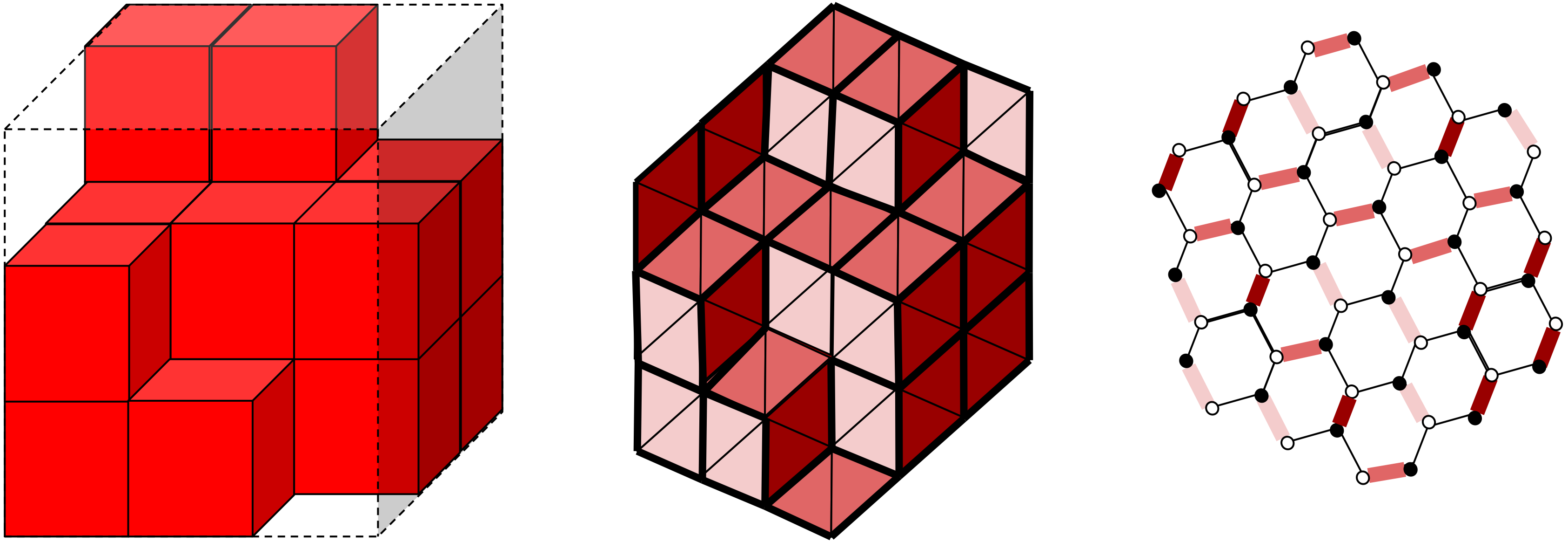
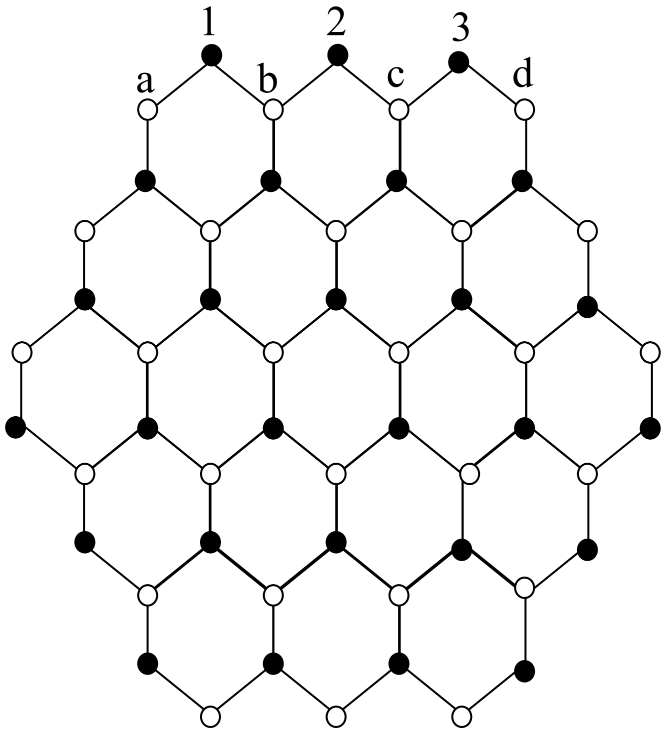
Plane Partitions



Adjacency Matrices



$$
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ \\
\end{array}
\begin{array}{c}
\begin{matrix} a & b & c & d & \end{matrix} \\
\begin{pmatrix}
1 & 1 & 0 & 0 & \cdots \\
0 & 1 & 1 & 0 & \cdots \\
0 & 0 & 1 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\end{array}
$$

Round Answers

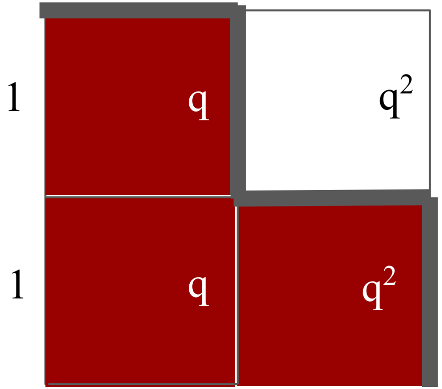
- An answer is *round* if it is a product of small factors
  - **Ex:** The number of subsets with k elements from a set with n elements:

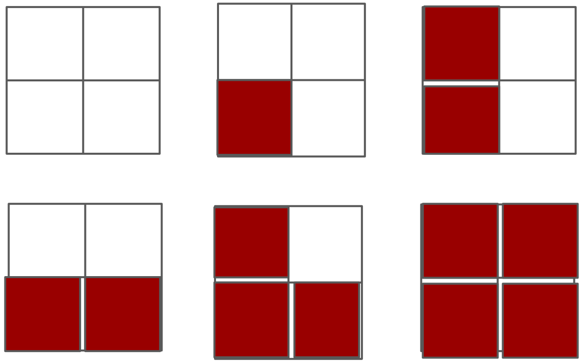
$$\binom{n}{k} = \frac{(n)(n-1)...(n-(k-1))}{(k)(k-1)...(1)}$$

  - **Ex:** The number of plane partitions in an a x b x c dimensional box:

$$\frac{H(a+b+c)H(a)H(b)H(c)}{H(a+b)H(a+c)H(b+c)} \quad \text{where} \quad H(n) = (n-1)!(n-2)!...1!$$

q-Analogs





How many different ways can we partition this region?

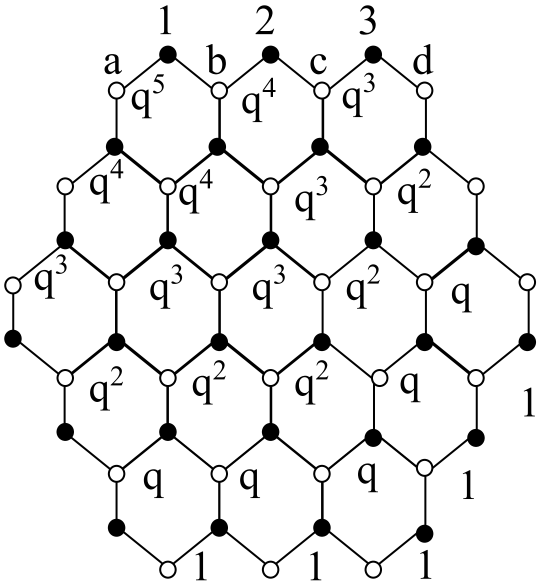
$$\binom{4}{2} = 6$$

What if we wanted to keep track of the number of boxes in each partition?

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q[n-1]_q \ldots [n-(k-1)]_q}{[k]_q[k-1]_q \ldots [1]_q} \qquad [n]_q = \frac{1-q^n}{1-q}$$

q-Carlitz Matrices



$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \\ \end{array}
\begin{array}{c}
\begin{array}{ccccc} a & b & c & d & \end{array} \\
\begin{pmatrix}
q^5 & 1 & 0 & 0 & \cdots \\
0 & q^4 & 1 & 0 & \cdots \\
0 & 0 & q^3 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\end{array}
$$

$$
C(a,b,c;q) = \left(q^{(c-i-1)j} \begin{bmatrix} a \\ b+i-j \end{bmatrix}_q \right)
$$

Smith Normal Forms & Cokernels

- Let A be a matrix with entries in a ring, R. A admits a Smith Normal Form if there exists matrices B,C in GL(R) such that BAC is a diagonal matrix with diagonal entries $(a_1,a_2,\ldots,a_k)$ with the property that $a_i | a_{i+1}$ for all i
- Let A be an n by k matrix with entries in a ring, R. A can be viewed as a homomorphism from R^k into R^n. The cokernel of A is defined to be $R^n/\text{im}(A)$
  - **Ex:** $A = \begin{pmatrix} 4 & 6 \\ 6 & 4 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 1 \\ 1 & -4 \end{pmatrix}$ $C = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$

    $BAC = \begin{pmatrix} 2 & 0 \\ 0 & 10 \end{pmatrix}$ $\quad coker(A) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/10$

Conjectures

- C(a,b,c;q) admits a Smith Normal Form over $\mathbb{Z}[q, q^{-1}]$
- The entries of the SNF of C(a,b,c;q) are squarefree over $\mathbb{Z}[q, q^{-1}]$
- # non unit entries in the SNF of C(a,b,c;q) = min(a,b,c) over $\mathbb{Z}[q, q^{-1}]$
- # non unit entries in the SNF of C(a,a,a;1)=a if and only if a=2^k over $\mathbb{Z}$
- $\begin{bmatrix} n \\ k \end{bmatrix}_q x + \begin{bmatrix} n \\ k+1 \end{bmatrix}_q y = gcd(\begin{bmatrix} n \\ k \end{bmatrix}_q, \begin{bmatrix} n \\ k+1 \end{bmatrix}_q)$ has solutions in $\mathbb{Z}[q, q^{-1}]$

Some things I have found

- If (k+1)(k+2) divides (n-k) or (n+1), then $\begin{bmatrix} n \\ k \end{bmatrix}_q \Big| \begin{bmatrix} n \\ k+1 \end{bmatrix}_q, \begin{bmatrix} n \\ k+2 \end{bmatrix}_q$
  - Thus, for such n and k, C(n,k,2;q) admits a Smith Normal Form over $\mathbb{Z}[q, q^{-1}]$
- $gcd(\begin{bmatrix} n \\ k \end{bmatrix}_q, \begin{bmatrix} n \\ k+1 \end{bmatrix}_1) = \frac{[gcd(n+1, k+1)]}{[k+1]} \begin{bmatrix} n \\ k \end{bmatrix}_q$
- If gcd(k+1,n+1)=1 and (k+1)|n, n+2, n-k or if gcd(k+1,n+1)=1 and n=2k, then $\begin{bmatrix} n \\ k \end{bmatrix}_q x + \begin{bmatrix} n \\ k+1 \end{bmatrix}_q y = gcd(\begin{bmatrix} n \\ k \end{bmatrix}_q, \begin{bmatrix} n \\ k+1 \end{bmatrix}_q)$ has solutions in $\mathbb{Z}[q, q^{-1}]$

Thank You!