Nico's Numerical Nightmare: An upper bound on p for disconnected G_p

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The Cage

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Theorem

If p is greater than 10^{1310}, then G_p is connected.

Order of an Element

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The *order of a triple* (x, y, z) is the maximum of the order of its elements x, y, and z.

Order of an Element (Continued)

Lemma

If $x = h + h^{-1}$ with $h \in \mathbb{F}_p$ or $h \in \mathbb{F}_{p^2}$, then the order of x is the order of h in $\mathbb{F}_p^*$ or $\mathbb{F}_{p^2}^*$ (BGS)

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$h \in \mathbb{F}_p$ Let $p = 7$ and $x = 6$. We want to find h such that $h + h^{-1} = 6$.
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$h = \frac{6 \pm \sqrt{36-4}}{2} = \frac{6 \pm 2}{2} = 4, 2$.
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Similar to above we get $h = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm \sqrt{5}}{2}$. Since 5 is not a quadratic residue mod 7, this is in $\mathbb{F}_{p^2}$ but is still a solution to $h + h^{-1} = 4$

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Proof.

Since $x \in \mathbb{F}_p^*$, $x^{p-1} = 1$, we have $x^p = (h + h^{-1})^p = h^p + h^{-p} = x$.
Since h and h^{-1} are the only solutions to $x = y + y^{-1}$, we see $h^p = h$ or $h^p = h^{-1}$.
So, $h^{p+1} = 1$ or $h^{p-1} = 1$ giving the desired result. □

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Lemma

All elements of the triples in the orbit of (x, y, z) *with respect to* x *are of the form* $h + \frac{\sigma}{h}$ *where* $h + h^{-1} = x$ *and* $\sigma \neq 1$ *(BGS)*

The End Game

Definition (The End Game)

We are in *The End Game* when a triple has order greater than $p^{1/2+\delta}$ for $\delta > 0$.

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Theorem

All triples in the end game have a triple of maximal order in their orbits (BGS).

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Definition (The Middle Game)

We are in *The Middle Game* when a triple has order less than $p^{1/2+\delta}$.

An Important Result

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Every component of G_p has a size divisible by p (Chen).

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So, we do not have to show every triple is connected to the cage, only that all but at most $p - 1$ are.

Our goal is to show for what values of p we can guarantee this for.

Connecting to Triple of Larger Order

Let a triple (x_1, x_2, x_3) have order $n < p^{1/2+\delta}$ with x_1 being the coordinate of largest order.

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Example

If a triple (x_1, x_2, x_3) has order 12 and we know there are 10 triples in the orbit of order at most 12, there must be a triple of order greater than 12.

What to Bound

Theorem

Let $x_1 = h_1 + h_1^{-1}$ with the order of x_1 being n. The number of elements in the orbit of (x_1, x_2, x_3) of the form $h_2 + h_2^{-1}$ where $ord(h_2) = m \leq n$ is at most

$$20 \cdot \max\{(n \cdot m)^{\frac{1}{3}}, \frac{(n \cdot m)}{p}\}$$

(Corvaja and Zannier)

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$$20 \cdot \max\{(n \cdot m)^{\frac{1}{3}}, \frac{(n \cdot m)}{p}\}$$

(Corvaja and Zannier)

Since n and m in the middle game are less than $p^{1/2+\delta}$ the second part is irrelevant so we get an upper bound of $20 \cdot (n \cdot m)^{\frac{1}{3}}$

Summation we are After

To show the triple of order y as an element of larger order in its orbit, it suffices to show

$$20 \cdot n^{\frac{1}{3}} \sum_{\substack{d \leq n \\ d \mid p \pm 1}} d^{\frac{1}{3}} < n$$

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Unfortunately, this doesn't always hold, so we want to find the smallest value of n where it fails. This will depend on the prime p.

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It suffices to show $20 \cdot C \cdot p^{\frac{1}{\log\log(p)}} \cdot n^{\frac{2}{3}} < n$ or $(20 \cdot C \cdot p^{\frac{1}{\log\log(p)}})^3 < n$.

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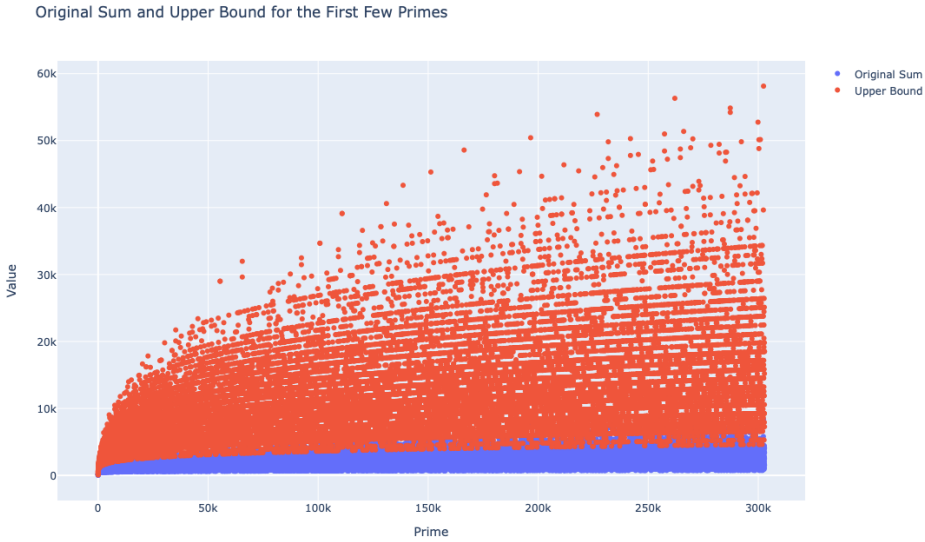
Notice if we just had the order of a triple divides $p^2 - 1$ instead of $p \pm 1$ we would have roughly the square of this for our answer.

Graph for Upper Bound



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So, let $y_p \leq (20 \cdot C \cdot p^{\frac{1}{\log\log(p)}})^3$ be the greatest divisor with this property.

Another Summation

Let $y_p \leq (20 \cdot C \cdot p^{\frac{1}{\log \log(p)}})^3$. Triples with order less than y_p may not be connected to the cage.

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How many such triples are there? For the first coordinate, there are $\sum_{\substack{d \leq y_p \\ d \mid p \pm 1}} d$ possibilities, since each divisor d contributes no more than d elements of that order.

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This is bounded by

$$\sum_{\substack{d \leq y_p \\ d \mid p \pm 1}} d < C \cdot y_p \cdot \tau(p \pm 1) < C' \cdot p^{\frac{4}{\log\log(p)}}$$

where C' is a "small" constant.

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To see this, set x and y in the Markoff equation $x^2 + y^2 + z^2 - xyz = 0$
This becomes a quadratic equation in z and we see the solutions are
$z = \frac{xy \pm \sqrt{(xy)^2 - 4(x^2+y^2)}}{2}$.

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In total, there are at most $2 \cdot (C' \cdot p^{\frac{4}{\log\log(p)}})^2 = C'' p^{\frac{8}{\log\log(p)}}$ triples where C'' is again "small".

The Big Reveal

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This, with the theorem of Chen from earlier, shows if $p > e^{e^8} \approx 10^{1300}$ then G_p is connected.

Thank You!