Effects of viscoelasticity on a two-link filament model

Michaela Rapier and Sophia Nelson

August 12, 2021

What do we want to know?

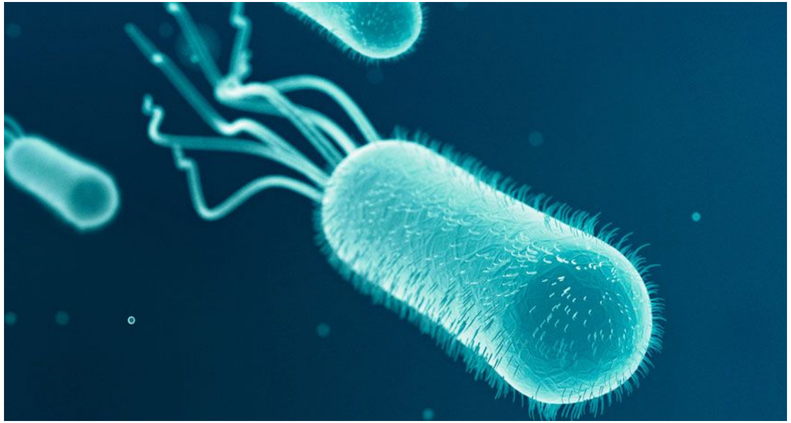
“How does fluid elasticity affect the flagellar beat of microscopic swimmers?”

What does it mean to swim?

"...You are put in some liquid and are allowed to deform your body in some manner." -Purcell (1976)

Cyclic deformation (strokes) is optimal to keep swimming.

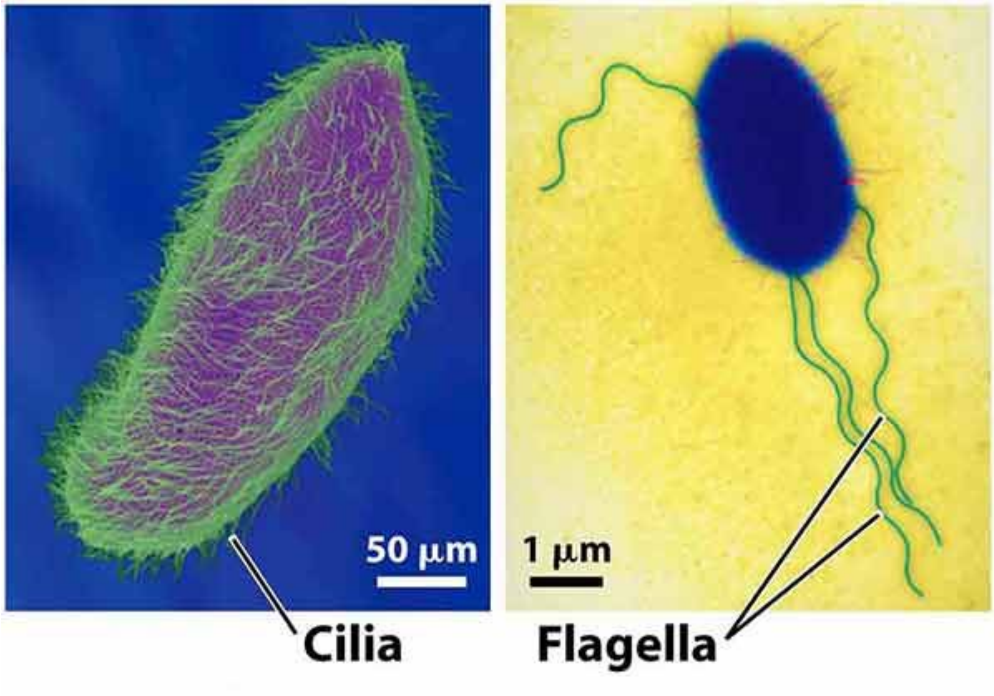
Macroscopic Scale



Microscopic Scale

635837344808987811-PTH1122-STATE-SWIM-MEET-12.JPG (3200×1680) (gannett-cdn.com)
860-swimming-bac-header-iStock_000022122825_Medium.jpg (860×460) (sciencenewsforstudents.org)

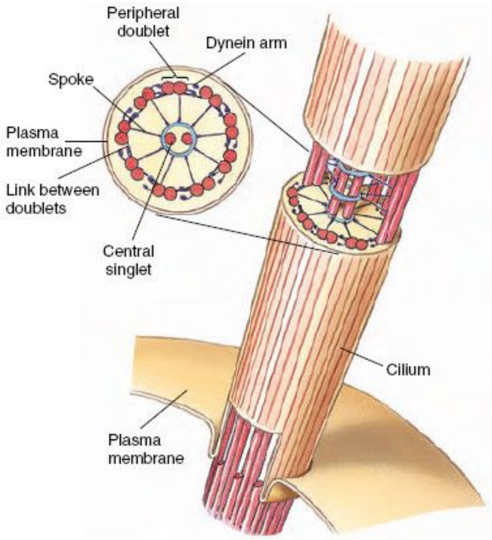
Flagella and Cilia

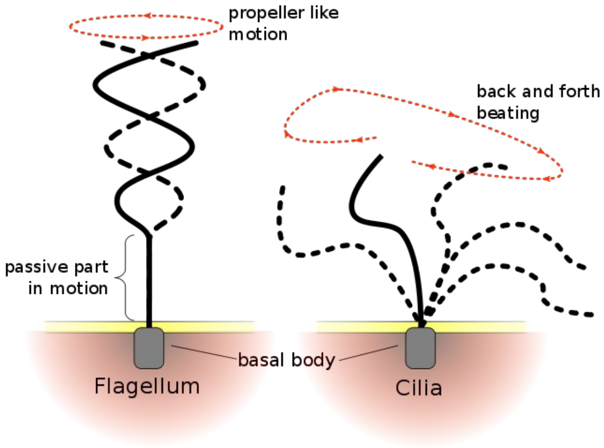


Flagella and cilia are tail or hairlike filaments on microscopic swimming organisms. They propel the organisms through fluid by generating beats.

[Differences-Between-Cilia-and-Flagella.jpg (600×421) (microbiologyinfo.com)]

Flagellar Beats

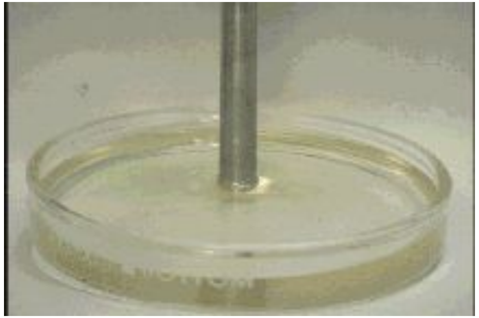




Dynein motors along the flagella/cilia cause them to bend. “Beats” result from coupled reactions from the surrounding fluid, elastic forces of the bending filament, and molecular motor activity.

[fig011.jpg (473×415) (biocyclopedia.com)](fig011.jpg)
https://upload.wikimedia.org/wikipedia/commons/a/a7/Flagellum-beating.svg

Complex Fluids - Fluid Elasticity



Rod Climbing

Shear Thickening

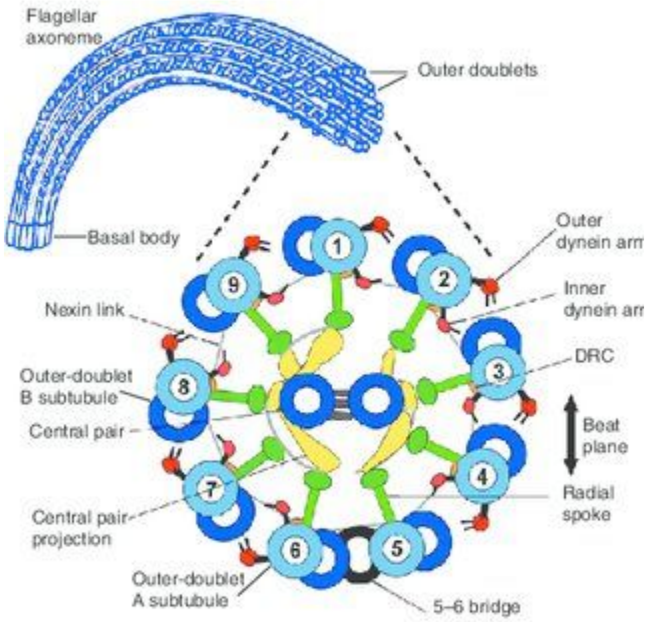
Complex fluids have a nonlinear relationship between stress and strain. So, they don’t behave like Newtonian fluids.

- Often mixtures; have polymers
- Viscosity isn’t constant
- Normal stresses from shearing
- Elastic recoil
- Yield Stress

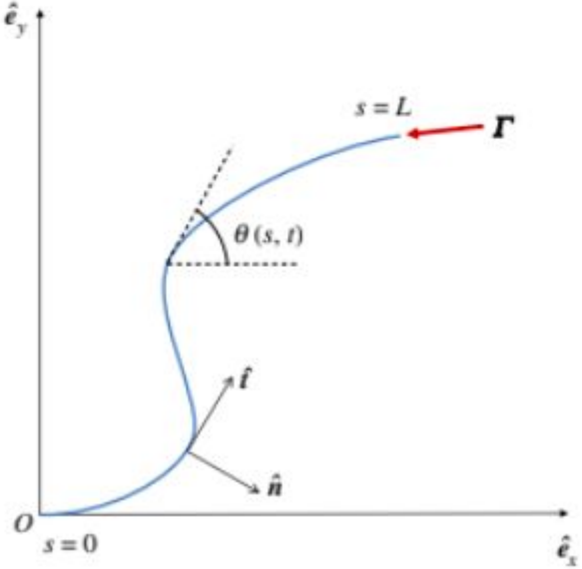
[rod_climb_clip.gif (216×144) (mit.edu)](rod_climb_clip.gif)
[shapeimage_1.png (700×396) (uchicago.edu)](shapeimage_1.png)

But modeling all these affects to the movements of an elastic filament is tricky…
So, we considered a simpler model.

Follower Force Model in a Viscous Fluid

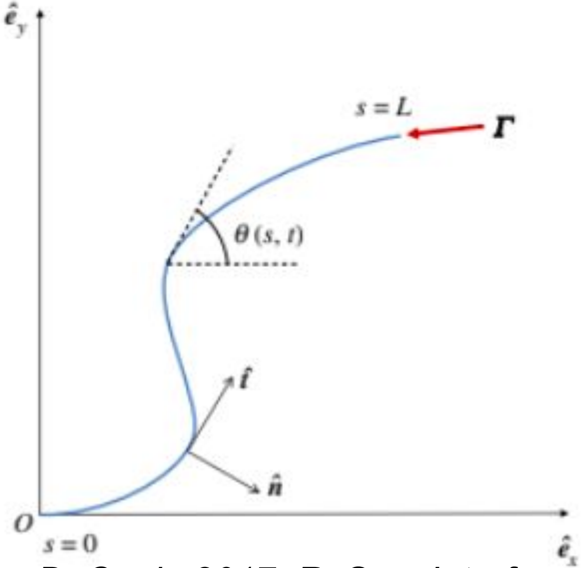


Motion of a flagella is influenced by a number of aspects due to its complex internal structure, we can simplify this by considering a tangential, follower force that is acting on the tip of the flagella.

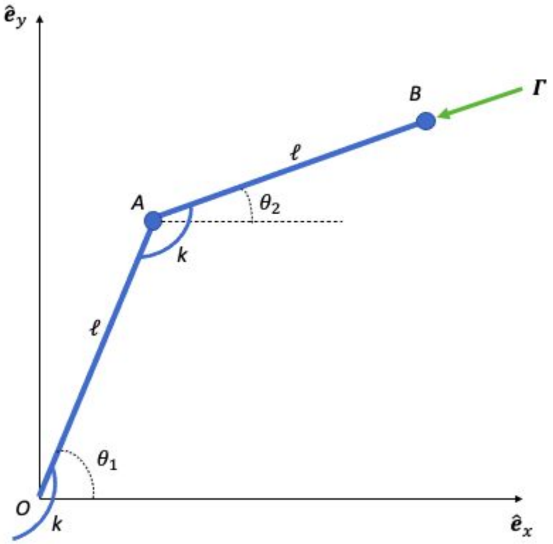


https://www.researchgate.net/profile/Charles-Lindemann/publication/41419396/figure/fig1/AS:340639242833932@1458226132687/Schematic-diagram-of-the-flagellar-axoneme-in-cross-section-Structures-that-are_Q320.jpg

Equations of Motion for continuous vs two-link model



DeCanio 2017, R. Soc. Interface



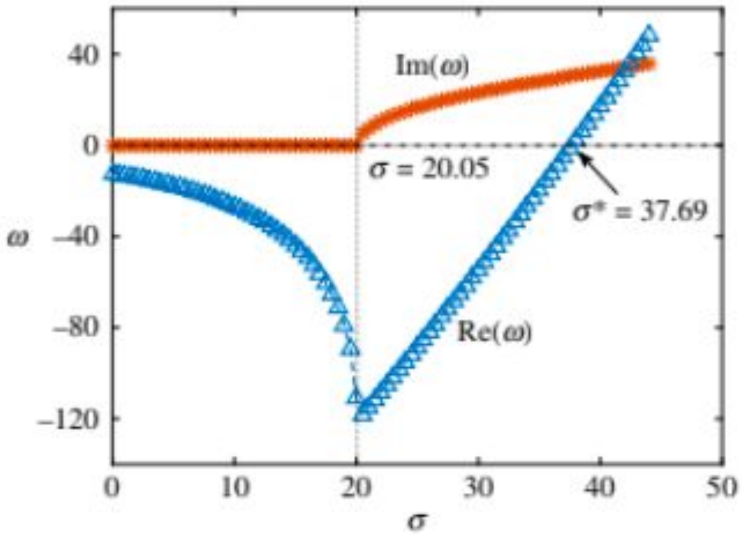
PDEs with boundary conditions

$$\theta_t = -\theta_{ssss} - [\Lambda - 3(\eta+1)\theta_s^2]\theta_{ss} - (\eta+1)\Lambda_s\theta_s$$
$$\Lambda_{ss} - \eta^{-1}\theta_s^2\Lambda = -\eta^{-1}\theta_s^4 + 3\theta_{ss}^2 + (3+\eta^{-1})\theta_s\theta_{sss}$$
$$\theta_{sss}(0,t) - \theta_s(0,t)^3 + \theta_s(0,t)\Lambda(0,t) = 0$$
$$\Lambda_s(0,t) - 3\theta_s(0,t)\theta_{ss}(0,t) = 0$$

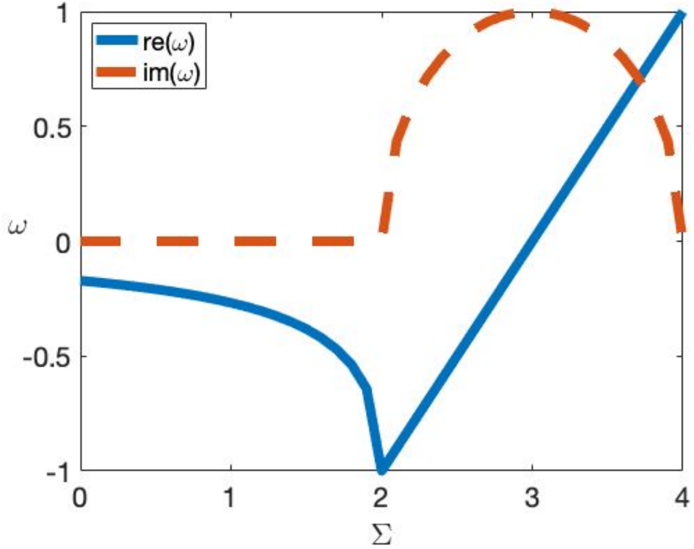
ODEs with initial conditions

$$\Sigma\sin(\theta_1 - \theta_2) - [2\dot{\theta}_1 + \dot{\theta}_2\cos(\theta_1 - \theta_2)] - 2\theta_1 + \theta_2 = 0$$
$$-\dot{\theta}_1\cos(\theta_1 - \theta_2) - \dot{\theta}_2 + \theta_1 - \theta_2 = 0$$

Dynamics are similar for two-link model



DeCanio 2017, R. Soc. Interface

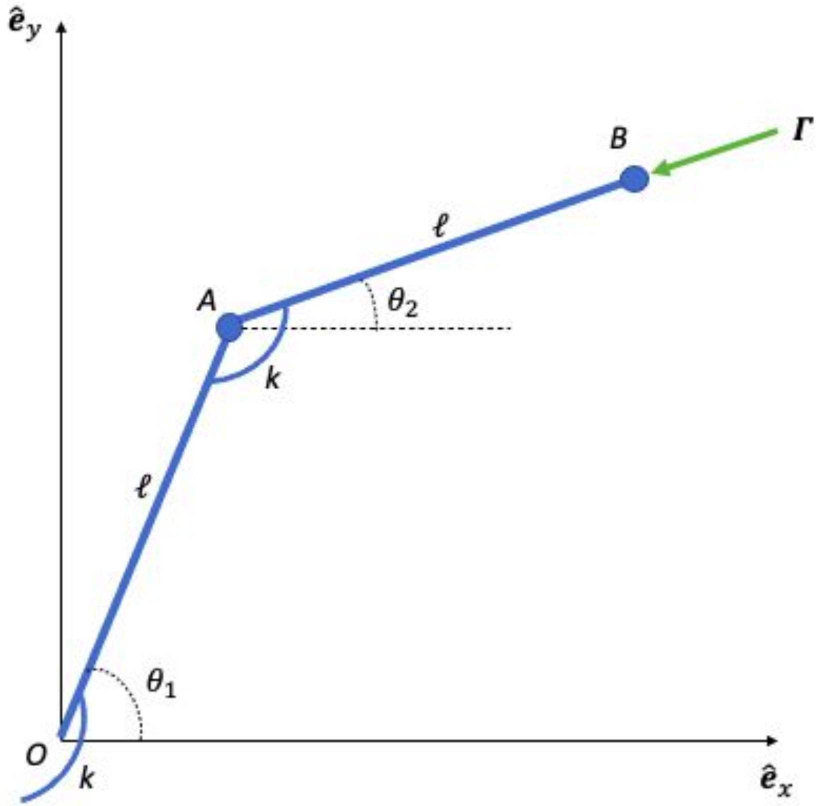


σ and Σ - ratio between strength of follower force and elastic force

Real part of ω represents growth rate, imaginary part of ω represents frequency

=>Higher order and lower order modeling of the filament yield similar dynamics, so we will continue with the two link model to understand the effect of viscoelasticity

Two-link Model



O	Clamped end
A	Joined links
B	Free end
l	Link Length
k	Torsional Spring Strength
θ_1	Degree of freedom about O
θ_2	Degree of freedom about A
Γ	Tangential Follower Force

Variables of Two-link Model

Locations of points A and B:

$$r_A = A - O = l(\cos\theta_1, \sin\theta_1)$$
$$r_B = B - O = l(\cos\theta_1 + \cos\theta_2, \sin\theta_1 + \sin\theta_2)$$

Velocities of points A and B:

$$v_A = \dot{r}_A = l\dot{\theta}_1(-\sin\theta_1, \cos\theta_2)$$
$$v_B = \dot{r}_B = l[\dot{\theta}_1(-\sin\theta_1, \cos\theta_1) + \dot{\theta}_2(-\sin\theta_2, \cos\theta_1)]$$

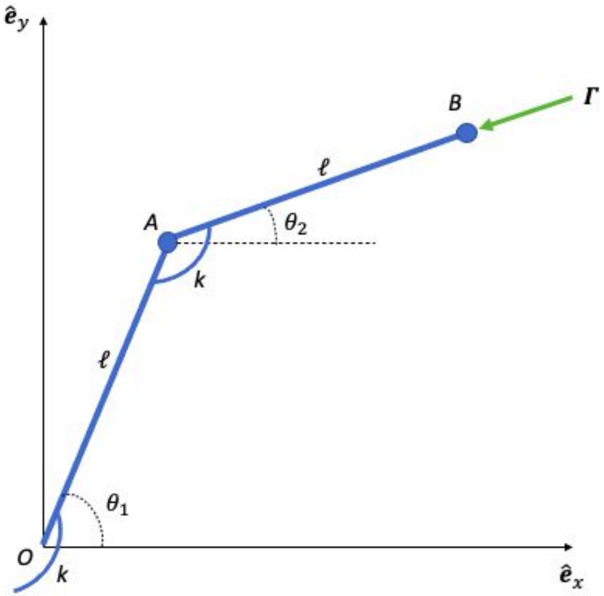
Follower Force:

$\mathbf{\Gamma} = -\Gamma\hat{t}$ where $\Gamma > 0$ is the magnitude and $\hat{t} = (\cos\theta_2, \sin\theta_2)$ is the unit tangent vector that joins points A and B.

Torsional Springs Restoring Moments:

$M_O = -k\theta_1$ at point O

$M_A = -k(\theta_2 - \theta_1)$ at point A



With this set up in mind, we

- Derived equations of motion for the two-link filament in both viscous (DeCanio) and viscoelastic fluids
- Numerically simulated the nonlinear systems
- Ran linear stability analysis and
- Tried to find the expected value of the frequency of filament oscillations in the viscoelastic case

To find “How does viscoelasticity affect the oscillations of a two-link model?”

Viscous Equations of Motion with Nondimensionalization

Applied the principle of virtual work:

$$\Gamma \cdot \delta r_B + F_B \cdot \delta r_B + F_A \cdot \delta r_A - k\theta_1\delta\theta_1 - k(\theta_1 - \theta_2)(\delta\theta_1 - \delta\theta_2) = 0$$

Fluid viscous forces = drag forces: $F_A = -\zeta v_A,\ F_B = -\zeta v_B.$

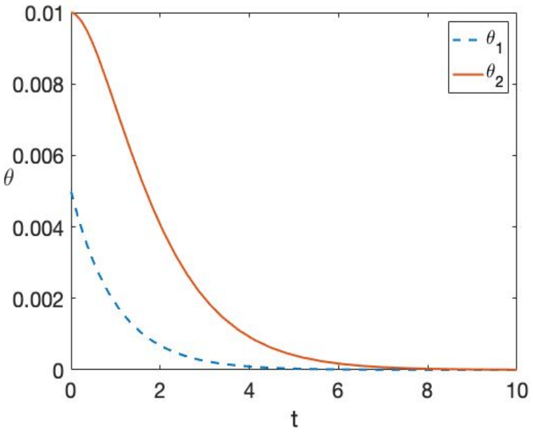
Nondimensional scaling: $\Sigma = \dfrac{\Gamma l}{k},\ \hat{t} = \dfrac{kt}{\zeta l^2}.$

Separating the arbitrary $\delta\theta_1$ and $\delta\theta_2$ yielded the nonlinear system

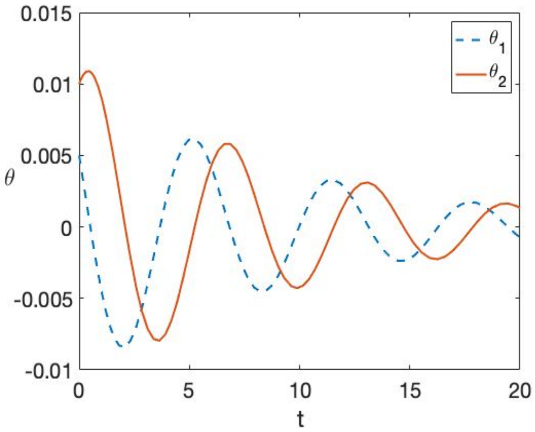
$$\Sigma \sin(\theta_1 - \theta_2) - [2\dot{\theta}_1 + \dot{\theta}_2 \cos(\theta_1 - \theta_2)] - 2\theta_1 + \theta_2 = 0$$

$$-\dot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_2 + \theta_1 - \theta_2 = 0$$

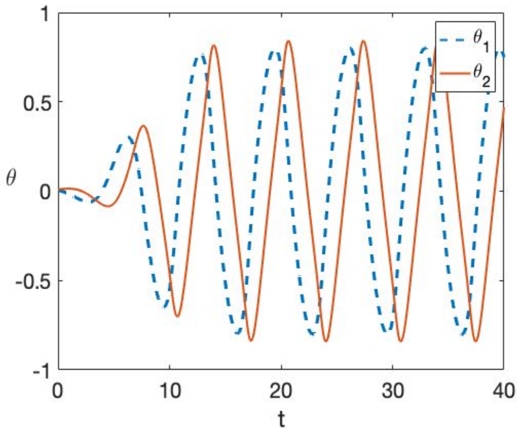
Viscous model solved in Matlab using ode45



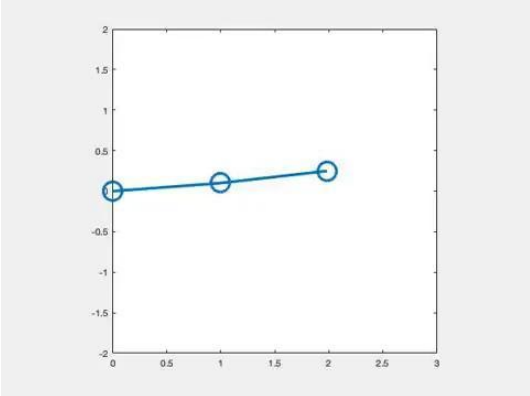
Σ=2



Σ=2.9



Σ=3.5



Linearization for viscous case

Linearized the system of equations about $\theta_1 = \theta_2 = 0$.

Applied the Taylor Series for $\sin(x)$ and $\cos(x)$ for when x is small:

$$\sin x = x \underbrace{- \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots}$$

= 0 because x is small

$$\cos x = 1 \overbrace{- \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots}$$

Linearized System of Equations:

1. $\Sigma(\theta_1 - \theta_2) - (2\dot{\theta}_1 + \dot{\theta}_2) - 2\theta_1 + \theta_2 = 0$

2. $-\dot{\theta}_1 - \dot{\theta}_2 + \theta_1 - \theta_2 = 0$

Linear Stability Analysis

Applied solution $\theta_i = \hat{\theta}_i e^{\omega \tilde{t}}$:

$$\Sigma(\hat{\theta}_1 - \hat{\theta}_2) - \omega(2\hat{\theta}_1 + \hat{\theta}_2) - 2\hat{\theta}_1 + \hat{\theta}_2 = 0$$

$$-\omega(\hat{\theta}_1 - \hat{\theta}_2) + \hat{\theta}_1 - \hat{\theta}_2 = 0$$

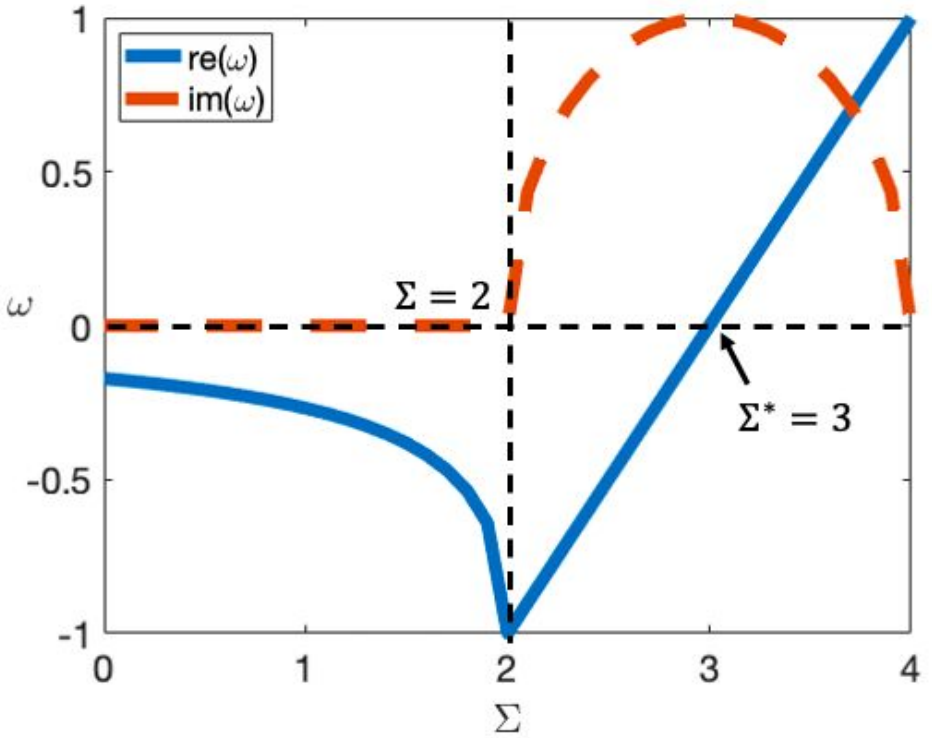
Separating the equations by $\hat{\theta}_1$ and $\hat{\theta}_2$ yielded the matrix system

$$\begin{bmatrix} (\Sigma - 2\omega - 2) & (-\Sigma - \omega + 1) \\ (-\omega + 1) & (-\omega - 1) \end{bmatrix} \begin{bmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

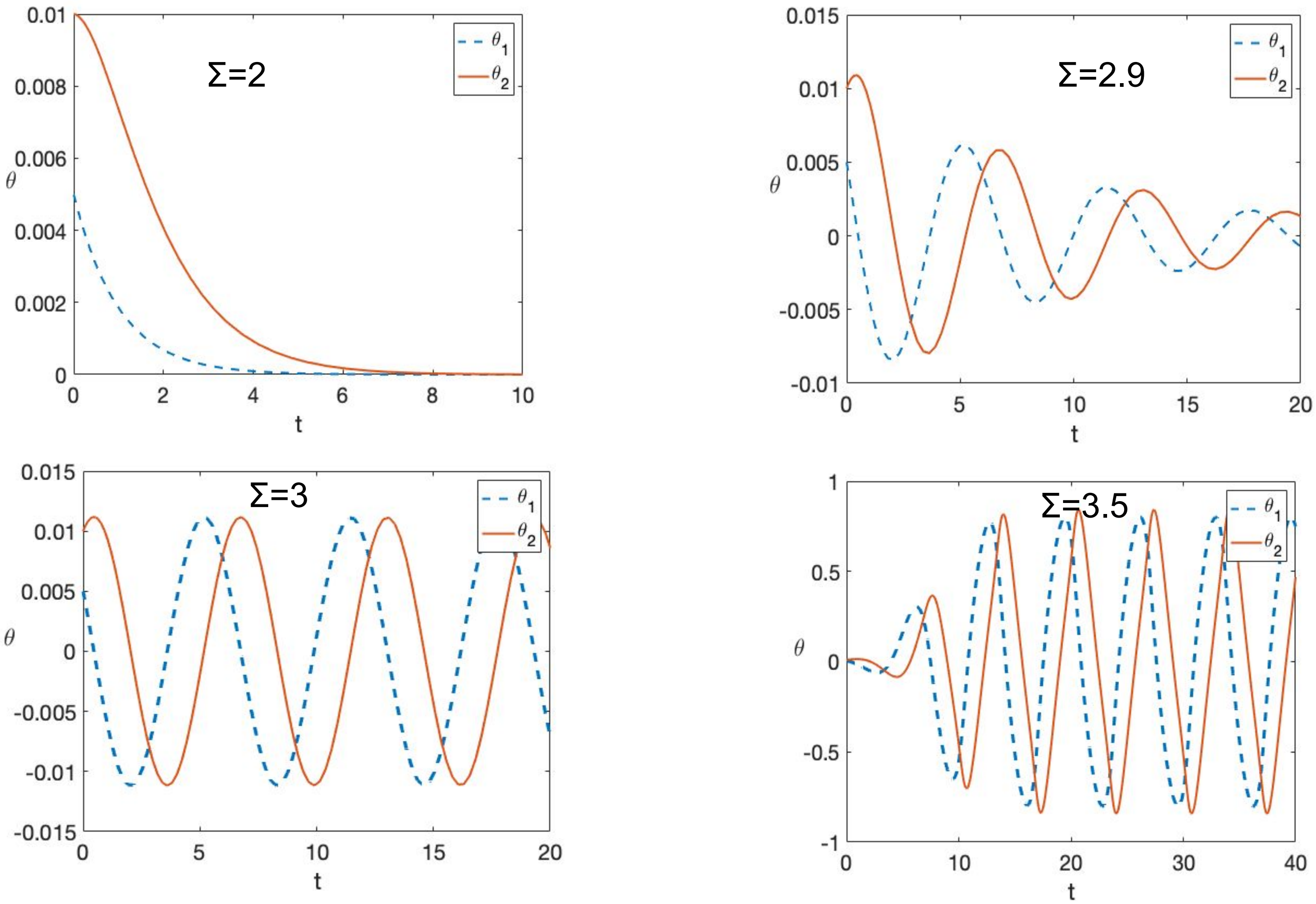
With the determinant zero, the solution for ω was

$$\omega_{\pm} = \Sigma - 3 \pm \sqrt{(\Sigma - 4)(\Sigma - 2)}$$

Stability Analysis results



1. $\Sigma \leq 2 \Rightarrow \omega_{\pm} < 0 \Rightarrow$ system is stable
2. $2 < \Sigma < 3 \Rightarrow$ Re($\omega_{\pm}$)<0 and Im($\omega_{\pm}$)≠0 ⇒ decaying oscillations
3. $\Sigma=3 \Rightarrow$ Re($\omega_{\pm}$)=0 and Im($\omega_{\pm}$)≠0 ⇒ system is stable with periodic oscillations
4. $3 < \Sigma < 4 \Rightarrow$ Re($\omega_{\pm}$)>0 and Im($\omega_{\pm}$)≠0 ⇒ exponentially growing oscillations
5. $\Sigma \geq 4 \Rightarrow \omega_{\pm} > 0 \Rightarrow$ system is unstable



Σ=2
Σ=2.9
Σ=3
Σ=3.5
θ_1
θ_2
θ
t

What is Hopf Bifurcation?

A textbook definition:

“The appearance or the disappearance of a periodic orbit through a local change in the stability properties of a steady point.”

Linear Stability

- Exponential decay stops and changes to exponential growth, and the oscillations change from stable to unstable when $Re(\omega\pm) = 0$, leaving constant oscillations of frequency $\frac{Im(\omega_{\pm})}{2\pi}$.

[Hopf bifurcation : definition of Hopf bifurcation and synonyms of Hopf bifurcation (English) (sensagent.com)]

Hopf Bifurcation: Viscous

Linear Analysis Solution:

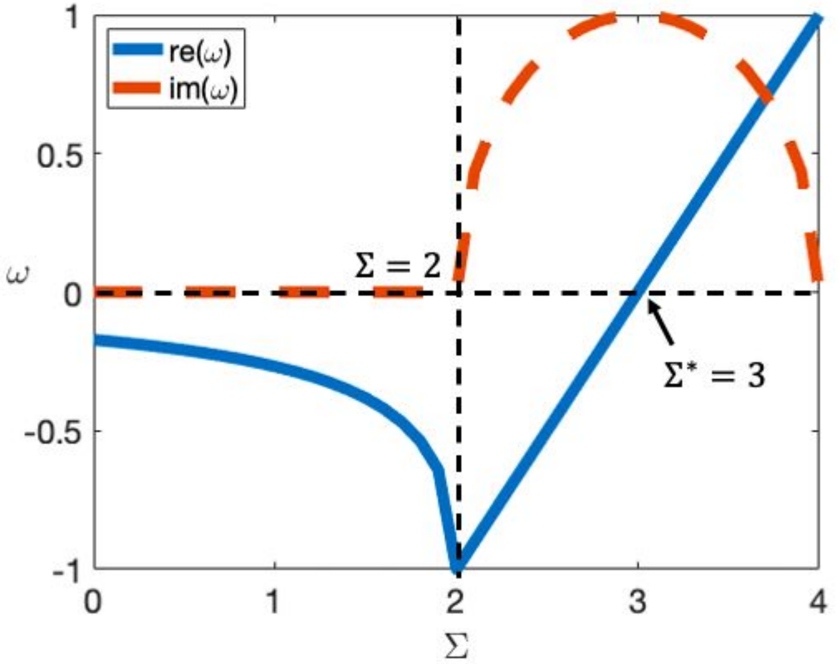
$$\omega_{\pm} = \Sigma - 3 \pm \sqrt{(\Sigma - 4)(\Sigma - 2)}$$

Real Part: $\Sigma - 3$

So,

$$\Sigma - 3 = 0$$

$$\Sigma = 3$$



Viscoelastic Fluid Properties

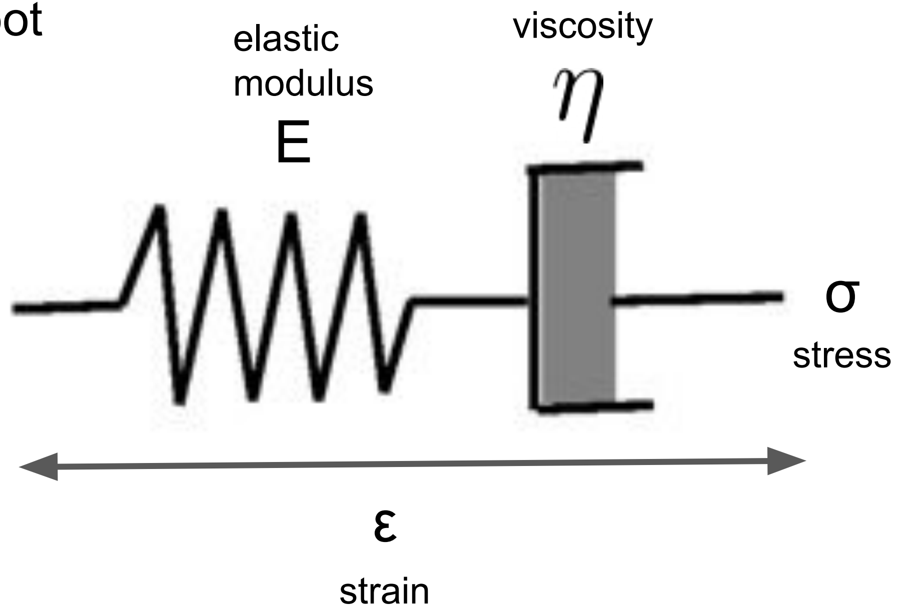
- Possess both fluid and solid properties
  - Viscosity- fluid property; measure of resistance to flow
  - Elasticity- solid property; ability to resume original shape after deformation
- Stress (σ) and is a function of strain (ε) and strain rate
  - $\sigma = \sigma(\epsilon, \dot{\epsilon})$

Maxwell Model

Linear arrangement of spring and dashpot

- Stress is equal throughout
  - $\sigma = \sigma_s = \sigma_d$
- Strain is additive
  - $\varepsilon = \varepsilon_s + \varepsilon_d$

$$\eta\dot{\sigma} + E\sigma = E\eta\dot{\epsilon}$$



Changing Drag Force

Maxwell Model:

$$\eta\dot{\sigma} + E\sigma = E\eta\dot{\epsilon} \Rightarrow \lambda\dot{\sigma} + \sigma = F_{viscous}$$

$\lambda = \dfrac{\eta}{E}$: the fluid relaxation time, $F_{viscous} = \eta\dot{\epsilon}$

Viscoelastic forces: $F_A \rightarrow \sigma_A$, $F_B \rightarrow \sigma_B$

$$\sigma_A = (\sigma_{Ax}, \sigma_{Ay}) \qquad \sigma_B = (\sigma_{Bx}, \sigma_{By})$$

Evolution of drag forces overtime:

$$\lambda\dot{\sigma}_A = -\zeta v_A - \sigma_A \qquad \lambda\dot{\sigma}_B = -\zeta v_B - \sigma_B$$

Viscoelastic equations of Motion

Principle of virtual work and fluid viscous forces:

$$\Gamma \cdot \delta r_B + \sigma_B \cdot \delta r_B + \sigma_A \cdot \delta r_A - k\theta_1\delta\theta_1 - k(\theta_1 - \theta_2)(\delta\theta_1 - \delta\theta_2) = 0$$
$$\lambda\dot{\sigma}_A + \sigma_A = -\eta v_A$$
$$\lambda\dot{\sigma}_B + \sigma_B = -\eta v_B$$

Initial nonlinear system:

$$-\Gamma l \sin(\theta_1 - \theta_2) + l[-(\sigma_{A_x} + \sigma_{B_x})\sin\theta_1 + (\sigma_{A_y} + \sigma_{B_y})\cos\theta_1] + k(\theta_2 - 2\theta_1) = 0$$
$$l[-\sigma_{B_x}\sin\theta_2 + \sigma_{B_y}\cos\theta_2] - k(\theta_2 - \theta_1) = 0$$
$$\lambda\dot{\sigma}_{A_x} + \sigma_{A_x} = \zeta l\dot{\theta}_1\sin\theta_1$$
$$\lambda\dot{\sigma}_{A_y} + \sigma_{A_y} = -\zeta l\dot{\theta}_1\cos\theta_1$$
$$\lambda\dot{\sigma}_{B_x} + \sigma_{B_x} = \zeta l(\dot{\theta}_1\sin\theta_1 + \dot{\theta}_2\sin\theta_2)$$
$$\lambda\dot{\sigma}_{B_y} + \sigma_{B_y} = -\zeta l(\dot{\theta}_1\cos\theta_1 + \dot{\theta}_2\cos\theta_2)$$

Linearization

Linearized about $\theta_1 = \theta_2 = 0$ with Taylor Series $\sin(x) = x$ and $\cos(x) = 1$

$$
\begin{aligned}
-\Gamma l(\theta_2 - \theta_1) + l(\sigma_{A_x} + \sigma_{B_x}) + k(\theta_2 - 2\theta_1) &= 0 \\
l\sigma_{B_y} - k(\theta_2 - \theta_1) &= 0 \\
\lambda\dot{\sigma}_{A_x} + \sigma_{A_x} &= 0 \\
\lambda\dot{\sigma}_{A_y} + \sigma_{A_y} &= -\zeta l\dot{\theta}_1 \\
\lambda\dot{\sigma}_{B_x} + \sigma_{B_x} &= 0 \\
\lambda\dot{\sigma}_{B_y} + \sigma_{B_y} &= -\zeta l(\dot{\theta}_1 + \dot{\theta}_2)
\end{aligned}
$$

Nondimensionalization

Scaling substitutions: $t = T\hat{t}$, $\sigma = \tilde{\Sigma}\hat{\sigma}$, and $\theta = \alpha\hat{\theta}$.

.

$$-\Gamma l\alpha(\hat{\theta}_2 - \hat{\theta}_1) + l\tilde{\Sigma}(\hat{\sigma}_{A_y} + \hat{\sigma}_{B_y}) + k\alpha(\hat{\theta}_1 - 2\hat{\theta}_2) = 0$$

$$l\tilde{\Sigma}\hat{\sigma}_{B_y} - k\alpha(\hat{\theta}_2 - \hat{\theta}_1) = 0$$

$$\lambda\tilde{\Sigma}\frac{1}{T}\dot{\hat{\sigma}}_{A_y} + \tilde{\Sigma}\hat{\sigma}_{A_y} = -\zeta l\alpha\frac{1}{T}\dot{\hat{\theta}}_1$$

$$\lambda\tilde{\Sigma}\frac{1}{T}\dot{\hat{\sigma}}_{B_y} + \tilde{\Sigma}\hat{\sigma}_{B_y} = -\zeta l\alpha\frac{1}{T}(\dot{\hat{\theta}}_1 + \dot{\hat{\theta}}_2)$$

Divided equations 1 and 2 by kα, 3 and 4 by $\tilde{\Sigma}$.

$$-\frac{\Gamma l}{k}(\hat{\theta}_2 - \hat{\theta}_1) + \frac{l\tilde{\Sigma}}{k\alpha}(\hat{\sigma}_{A_y} + \hat{\sigma}_{B_y}) + (\hat{\theta}_1 - 2\hat{\theta}_2) = 0$$

$$\frac{l\tilde{\Sigma}}{k\alpha}\hat{\sigma}_{B_y} - (\hat{\theta}_2 - \hat{\theta}_1) = 0$$

$$\lambda\frac{1}{T}\dot{\hat{\sigma}}_{A_y} + \hat{\sigma}_{A_y} = -\frac{\zeta l\alpha}{\tilde{\Sigma}}\frac{1}{T}\dot{\hat{\theta}}_1$$

$$\lambda\frac{1}{T}\dot{\hat{\sigma}}_{B_y} + \hat{\sigma}_{B_y} = -\frac{\zeta l\alpha}{\tilde{\Sigma}}\frac{1}{T}(\dot{\hat{\theta}}_1 + \dot{\hat{\theta}}_2)$$

Scaling Factors

Scaling factors: $T = \frac{\zeta l^2}{k}, \ \tilde{\Sigma} = \frac{k\alpha}{l}, \ \Sigma = \frac{\Gamma l}{k}, \ \Lambda = \frac{k\lambda}{\zeta l^2}$

Nondimensionalized linear system:

$$\Sigma(\hat{\theta}_1 - \hat{\theta}_2) + (\hat{\sigma}_{A_y} + \hat{\sigma}_{B_y}) + (\hat{\theta}_1 - 2\hat{\theta}_2) = 0$$

$$\hat{\sigma}_{B_y} - (\hat{\theta}_2 - \hat{\theta}_1) = 0$$

$$\Lambda\dot{\hat{\sigma}}_{A_y} + \hat{\sigma}_{A_y} = -\dot{\hat{\theta}}_1$$

$$\Lambda\dot{\hat{\sigma}}_{B_y} + \hat{\sigma}_{B_y} = -(\dot{\hat{\theta}}_1 + \dot{\hat{\theta}}_2)$$

Viscoelastic Linear Stability Analysis

Assumed solutions $\theta_j = \hat{\theta}_j e^{\omega \hat{t}}$ and $\sigma_j = \hat{\sigma}_j e^{\omega \hat{t}}$.

$$\Sigma(\hat{\theta}_1 - \hat{\theta}_2) + (\hat{\sigma}_{A_y} + \hat{\sigma}_{B_y}) + (\hat{\theta}_1 - 2\hat{\theta}_2) = 0$$

$$\hat{\sigma}_{B_y} - (\hat{\theta}_2 - \hat{\theta}_1) = 0$$

$$\Lambda\omega\hat{\sigma}_{A_y} + \hat{\sigma}_{A_y} = -\omega\hat{\theta}_1$$

$$\Lambda\omega\hat{\sigma}_{B_y} + \hat{\sigma}_{B_y} = -(\omega\hat{\theta}_1 + \omega\hat{\theta}_2)$$

Separating the equations by $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\sigma}_{A_y}$, and $\hat{\sigma}_{B_y}$ yielded the matrix system

$$\begin{bmatrix} (\Sigma - 2) & (-\Sigma + 1) & 1 & 1 \\ 1 & -1 & 0 & 1 \\ \omega & 0 & (\Lambda\omega + 1) & 0 \\ \omega & \omega & 0 & (\Lambda\omega + 1) \end{bmatrix} \begin{bmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\sigma}_{A_y} \\ \hat{\sigma}_{B_y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

With the determinant zero, the solution for ω was

$$\omega_\pm = \frac{\Sigma - \Lambda - 3 \pm \sqrt{(\Sigma - 4)(\Sigma - 2)}}{-2\Sigma\Lambda + \Lambda^2 + 6\Lambda + 1}$$

Hopf Bifurcation: Linear Relationship between Σ and Λ

Viscoelastic Linear Result:

$$\omega_{\pm} = \frac{\Sigma - \Lambda - 3 \pm \sqrt{(\Sigma - 4)(\Sigma - 2)}}{-2\Sigma\Lambda + \Lambda^2 + 6\Lambda + 1}$$

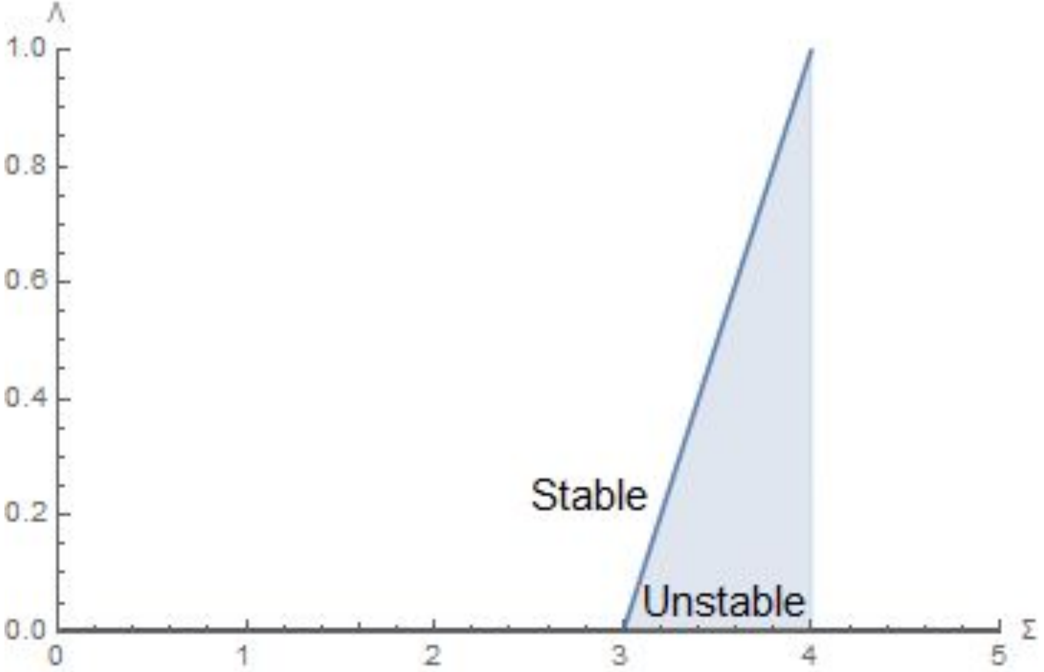
Real Part:

$$\Sigma - \Lambda - 3 = 0$$

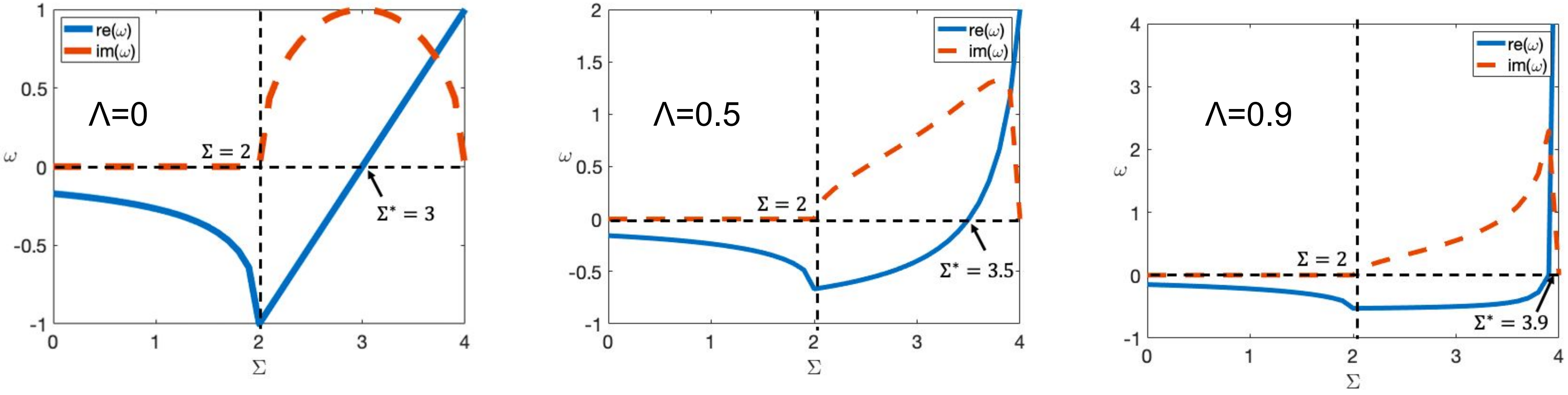
$$\Rightarrow \Sigma = \Lambda + 3$$

Boundaries:

$$3 < \Sigma < 4$$
$$0 < \Lambda < 1$$



Results of stability analysis are consistent with linear relationship



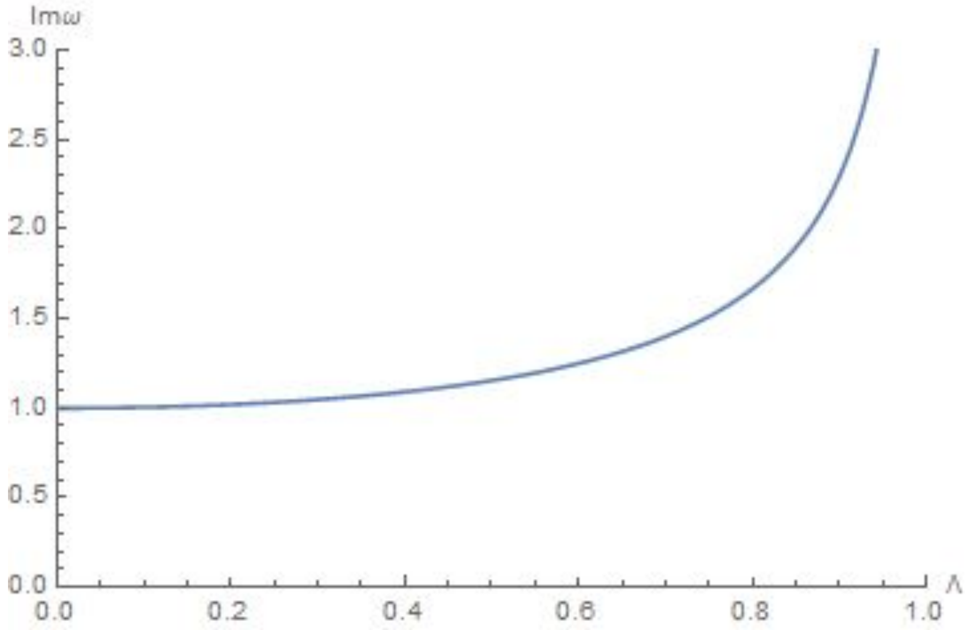
Predicting Frequency along Bifurcation Points

Substituting $\Sigma = \Lambda + 3$ in $\omega_{\pm}$ yielded

$$\omega_{\pm} = \frac{0 \pm \sqrt{(\Lambda - 1)(\Lambda + 1)}}{-\Lambda^2 + 1}$$

This predicts frequency of oscillations at the hopf bifurcation points with varying Λ:

*Graph for ω+



Model of Nonlinear System

System of DAEs- 4 DEs and 2 mechanical constraint equations

$$\Lambda\dot{\sigma}_{Ax} - \dot{\theta}_1 \sin\theta_1 = -\sigma_{Ax}$$
$$\Lambda\dot{\sigma}_{Ay} + \dot{\theta}_1 \cos\theta_1 = -\sigma_{Ay}$$
$$\Lambda\dot{\sigma}_{Bx} - \dot{\theta}_1 \sin\theta_1 - \dot{\theta}_2 \sin\theta_2 = -\sigma_{Bx}$$
$$\Lambda\dot{\sigma}_{By} + \dot{\theta}_1 \cos\theta_1 + \dot{\theta}_2 \cos\theta_2 = -\sigma_{By}$$
$$-\Sigma \sin(\theta_2 - \theta_1) - (\sigma_{A_x} + \sigma_{B_x}) \sin\theta_1 + (\sigma_{A_y} + \sigma_{B_y}) \cos\theta_1 + \theta_2 - 2\theta_1 = 0$$
$$-\sin\theta_2 \sigma_{B_x} + \cos\theta_2 \sigma_{B_y} - \theta_2 + \theta_1 = 0$$

What is a DAE?

Differential Algebraic System of Equations:

- Type of differential equations where one or more derivatives of dependent variables are not present in the equations.
- Variables are *algebraic* if they do not have their derivative in the equations, and the presence of algebraic variables means that you cannot translate the equations in the explicit form $y'=f(t,y)$

Matlab ODE Solver ODE45 could not solve our nonlinear equations.

DAE solvers: ODE15i and ODE15s.

[Solve Differential Algebraic Equations (DAEs) - MATLAB & Simulink (mathworks.com)]

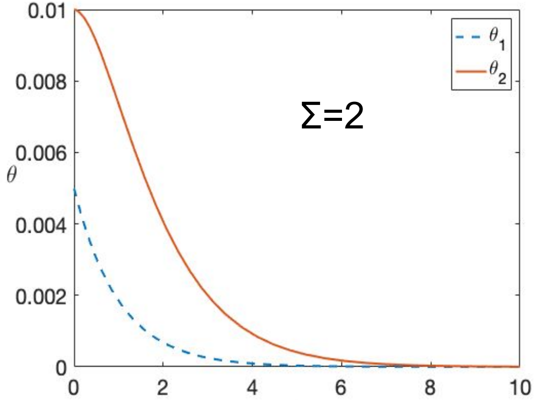
Mass matrix for nonlinear system

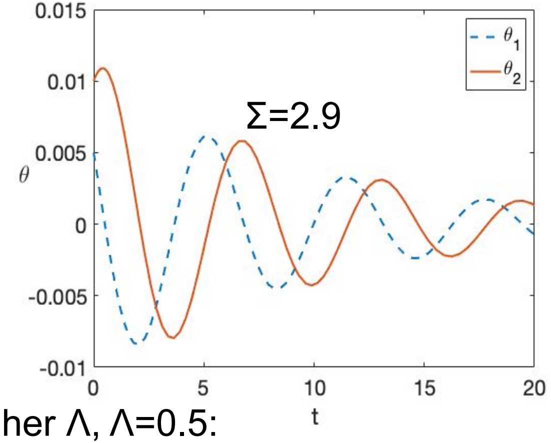
Need M such that My'=F(y, T)

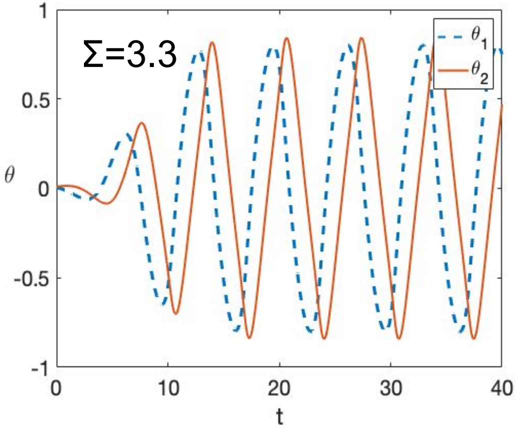
$$M = \begin{bmatrix} \Lambda & 0 & 0 & 0 & -\sin\theta_1 & 0 \\ 0 & \Lambda & 0 & 0 & \cos\theta_1 & 0 \\ 0 & 0 & \Lambda & 0 & -\sin\theta_1 & -\sin\theta_2 \\ 0 & 0 & 0 & \Lambda & \cos\theta_1 & \cos\theta_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad y' = \begin{bmatrix} \dot{\sigma}_{A_x} \\ \dot{\sigma}_{A_y} \\ \dot{\sigma}_{B_x} \\ \dot{\sigma}_{B_y} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

Viscoelastic model solved in Matlab using ode15s

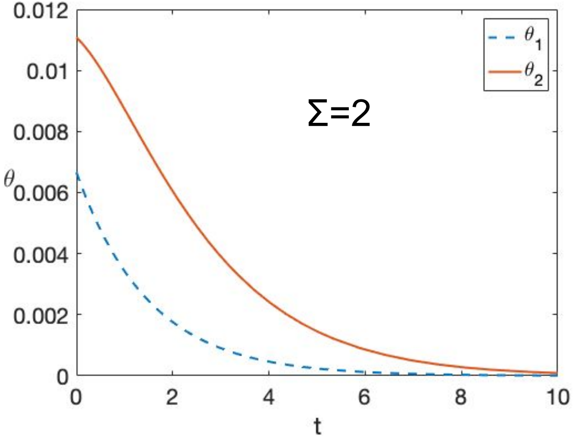
When Λ=0, we recover viscous model:

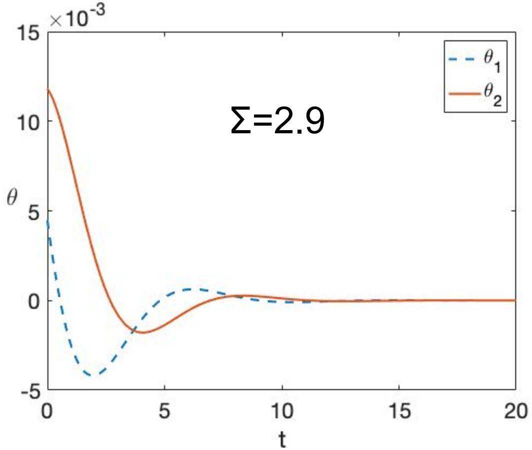


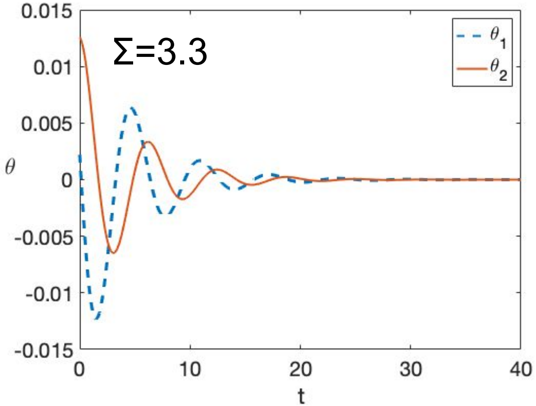




Shift in pattern of oscillations for higher Λ, Λ=0.5:

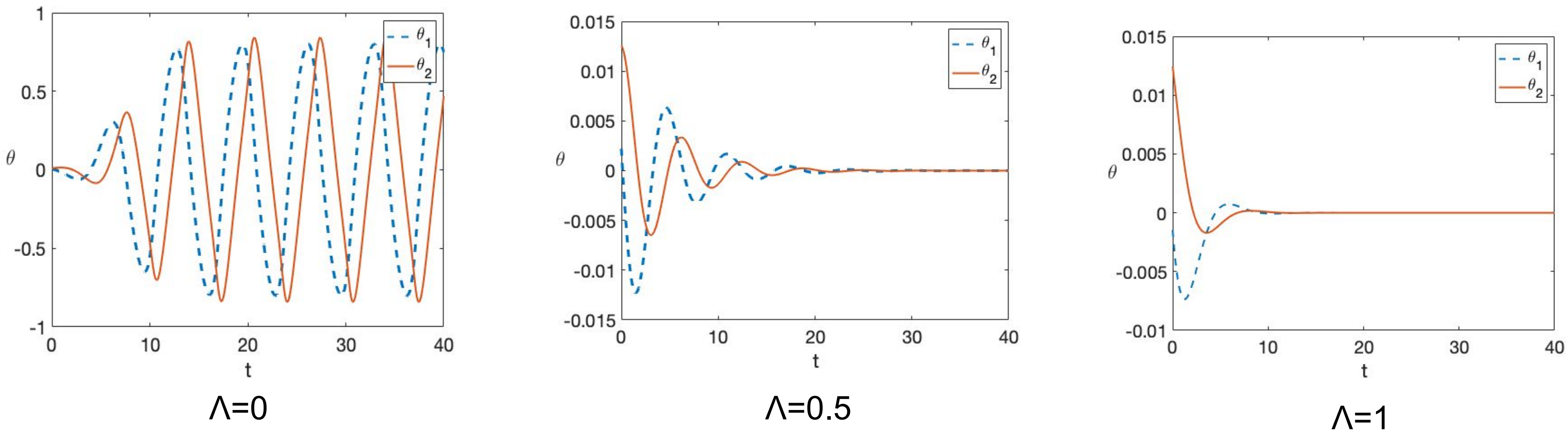






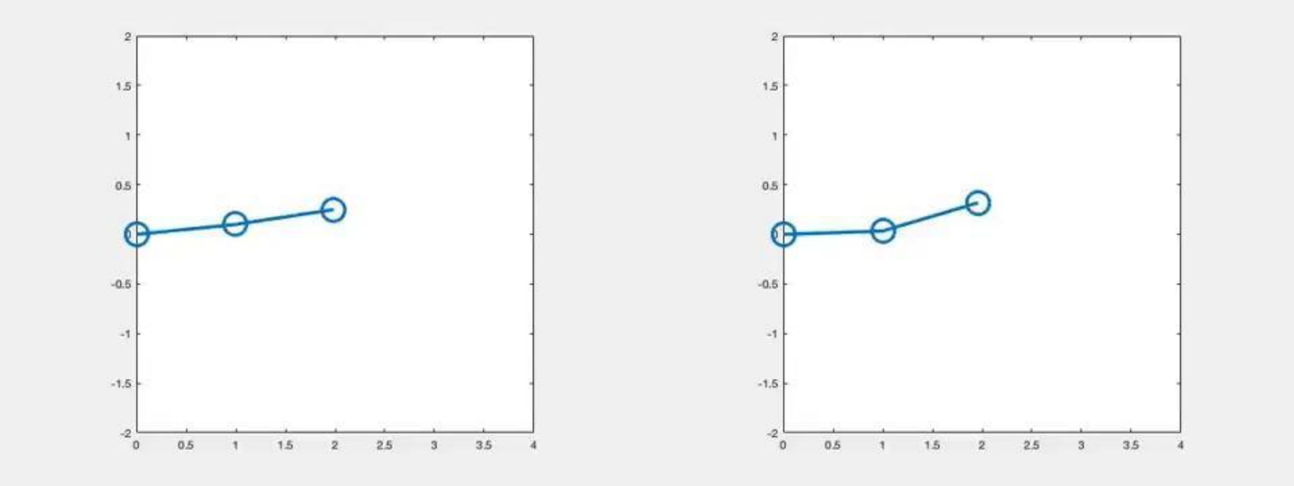
Fluid elasticity changes oscillation pattern for fixed Σ

Σ=3.3:



Fluid elasticity changes oscillation pattern for fixed Σ

Σ=3.3:



Λ=0 Λ=0.5 Λ=0.9

Issues with Maxwell Model

- Model breaks down for large Σ values
- We jump straight from a purely viscous to purely viscoelastic fluid
    - We want a way to smoothly transition between the two

… So we considered a new model!

Oldroyd-B Viscoelastic Model

Complex fluids have a total stress comprised of the fluid and polymer parts

$$\tau = \tau_f + \tau_p$$

With the Maxwell model, we considered just the viscosity of the polymers.

With this model, we consider both fluid and polymer viscosity:

$$F_B = -\zeta_f l[\dot{\theta}_1(-sin(\theta_1), cos(\theta_1)) + \dot{\theta}_2(-sin(\theta_2), cos(\theta_2))$$

$$\sigma_b = -\lambda((\dot{\sigma}_{B_x}, \dot{\sigma}_{B_y}) + \frac{\zeta_p}{\lambda}[\dot{\theta}_1(-sin(\theta_1), cos(\theta_1)) + \dot{\theta}_2(-sin(\theta_2), cos(\theta_2))$$

*Similar changes for F_A and σ_A.

Oldroyd-B Equations of Motion

Principle of virtual work and fluid viscous forces:

$$\Sigma \cdot \delta r_B + F_B \cdot \delta r_B + \sigma_B \cdot \delta r_B + F_A \cdot \delta r_A + \sigma_A \cdot \delta r_A - k\theta_1 \delta\theta_1 - k(\theta_1 - \theta_2)(\delta\theta_1 - \delta\theta_2) = 0$$

$$\lambda \dot{\sigma}_A + \sigma_A = F_A$$

$$\lambda \dot{\sigma}_B + \sigma_B = F_B$$

Initial nonlinear system:

$$-\Gamma l \sin(\theta_2 - \theta_1) + l(-\sin\theta_1(\sigma_{A_x} + \sigma_{B_x}) + \cos\theta_1(\sigma_{A_y} + \sigma_{B_y})) - 2\zeta_f l^2 \dot{\theta}_1 - \zeta_f l^2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) - 2k\theta_1 + k\theta_2 = 0$$

$$l(-\sin\theta_2 \sigma_{B_x} + \cos\theta_2 \sigma_{B_y}) - \zeta_f l^2(\dot{\theta}_2 + \dot{\theta}_1 \cos(\theta_1 - \theta_2)) - k(\theta_2 - \theta_1) = 0$$

$$\lambda \dot{\sigma}_{A_x} + \sigma_{A_x} = \zeta_p l \dot{\theta}_1 \sin\theta_1$$

$$\lambda \dot{\sigma}_{A_y} + \sigma_{A_y} = -\zeta_p l \dot{\theta}_1 \cos\theta_1$$

$$\lambda \dot{\sigma}_{B_x} + \sigma_{B_x} = \zeta_p l(\dot{\theta}_1 \sin\theta_1 + \dot{\theta}_2 \sin\theta_2)$$

$$\lambda \dot{\sigma}_{B_y} + \sigma_{B_y} = -\zeta_p l(\dot{\theta}_1 \cos\theta_1 + \dot{\theta}_2 \cos\theta_2)$$

Linearization

Assuming $\theta_1 = \theta_2 = 0, \cos(x) = 1$, and $\sin(x) = x$;

$$-\Gamma l(\theta_2 - \theta_1) + l(\sigma_{A_y} + \sigma_{B_y}) - 2\zeta_p l^2 \dot{\theta}_1 - \zeta_p l^2 \dot{\theta}_2 + k(\theta_2 - 2\theta_1) = 0$$

$$l\sigma_{B_y} - \zeta_p l^2 (\dot{\theta}_2 + \dot{\theta}_1) - k(\theta_2 - \theta_1) = 0$$

$$\lambda \dot{\sigma}_{A_y} + \sigma_{A_y} = -\zeta_p l \dot{\theta}_1$$

$$\lambda \dot{\sigma}_{B_y} + \sigma_{B_y} = -\zeta_p l (\dot{\theta}_1 + \dot{\theta}_2)$$

Nondimensionalization

Scaling factors: $t = T\hat{t}$, $\sigma = \tilde{\Sigma}\hat{\sigma}$, and $\theta = \alpha\hat{\theta} \rightarrow$

$$-\frac{\Gamma l}{k}(\hat{\theta}_2 - \hat{\theta}_1) + \frac{l\tilde{\Sigma}}{k\alpha}(\hat{\sigma}_{A_y} + \hat{\sigma}_{B_y}) - 2\frac{\zeta_p l^2}{kT}\dot{\hat{\theta}}_1 - \frac{\zeta_p l^2}{kT}\dot{\hat{\theta}}_2 + (\hat{\theta}_2 - 2\hat{\theta}_1) = 0$$

$$\frac{l\tilde{\Sigma}}{k\alpha}\hat{\sigma}_{B_y} - \frac{\zeta_p l^2}{kT}(\dot{\hat{\theta}}_2 + \dot{\hat{\theta}}_1) - (\hat{\theta}_2 - \hat{\theta}_1) = 0$$

$$\frac{\lambda}{T}\dot{\hat{\sigma}}_{A_y} + \hat{\sigma}_{A_y} = -\frac{\zeta_p l\alpha}{\tilde{\Sigma}T}\dot{\hat{\theta}}_1$$

$$\frac{\lambda}{T}\dot{\hat{\sigma}}_{B_y} + \hat{\sigma}_{B_y} = -\frac{\zeta_p l\alpha}{\tilde{\Sigma}T}(\dot{\hat{\theta}}_1 + \dot{\hat{\theta}}_2)$$

Linear System of Equations

Substitutions:

$$\tilde{\Sigma} = \frac{k\alpha}{l}, T = \frac{(\zeta_f + \zeta_p)l^2}{k}, \Sigma = \frac{\Gamma l}{k}, \Lambda = \frac{k\lambda}{\zeta_f l^2}, \beta = \frac{\zeta_p}{\zeta_f + \zeta_p}, (1-\beta) = \frac{\zeta_f}{\zeta_f + \zeta_p}$$

β = 0: No polymer viscosity → Viscous model

β = 1: Only polymer viscosity → Maxwell model

$$\Sigma(\theta_1 - \theta_2) + \sigma_{A_y} + \sigma_{B_y} - 2(1-\beta)\dot{\theta}_1 - (1-\beta)\dot{\theta}_2 + \theta_2 - 2\theta_1 = 0$$

$$\sigma_{B_y} - (1-\beta)(\dot{\theta}_2 + \dot{\theta}_1) - \theta_2 + \theta_1 = 0$$

$$\Lambda\dot{\sigma}_{A_y} + \sigma_{A_y} = -\beta\dot{\theta}_1$$

$$\Lambda\dot{\sigma}_{B_y} + \sigma_{B_y} = -\beta(\dot{\theta}_1 + \dot{\theta}_2)$$

Nonlinear System of Equations

$$\Sigma \sin(\theta_1 - \theta_2) - \sin\theta_1(\sigma_{A_x} + \sigma_{B_x}) + \cos\theta_1(\sigma_{A_y} + \sigma_{B_y}) - 2(1-\beta)\dot{\theta}_1 - (1-\beta)\dot{\theta}_2\cos(\theta_1 - \theta_2) - 2\theta_1 + \theta_2 = 0$$

$$-\sin\theta_2\sigma_{B_x} + \cos\theta_2\sigma_{B_y} - (1-\beta)(\dot{\theta}_2 + \dot{\theta}_1\cos(\theta_1 - \theta_2)) - \theta_2 + \theta_1 = 0$$

$$\Lambda\dot{\sigma}_{A_x} + \sigma_{A_x} = \beta\dot{\theta}_1\sin\theta_1$$

$$\Lambda\dot{\sigma}_{A_y} + \sigma_{A_y} = -\beta\dot{\theta}_1\cos\theta_1$$

$$\Lambda\dot{\sigma}_{B_x} + \sigma_{B_x} = \beta(\dot{\theta}_1\sin\theta_1 + \dot{\theta}_2\sin\theta_2)$$

$$\Lambda\dot{\sigma}_{B_y} + \sigma_{B_y} = -\beta(\dot{\theta}_1\cos\theta_1 + \dot{\theta}_2\cos\theta_2)$$

Linear Analysis

Same solutions $\theta_j = \hat{\theta}_j e^{\omega \hat{t}}$ and $\sigma_j = \hat{\sigma}_j e^{\omega \hat{t}}$.

$$\Sigma(\hat{\theta}_1 - \hat{\theta}_2) + \hat{\sigma}_{A_y} + \hat{\sigma}_{B_y} - \omega(1-\beta)(2\hat{\theta}_1 + \hat{\theta}_2) + \hat{\theta}_2 - 2\hat{\theta}_1 = 0$$

$$\hat{\sigma}_{B_y} - \omega(1-\beta)(\hat{\theta}_2 + \hat{\theta}_1) - \hat{\theta}_2 + \hat{\theta}_1 = 0$$

$$\omega\Lambda\hat{\sigma}_{A_y} + \hat{\sigma}_{A_y} + \omega\beta\hat{\theta}_1 = 0$$

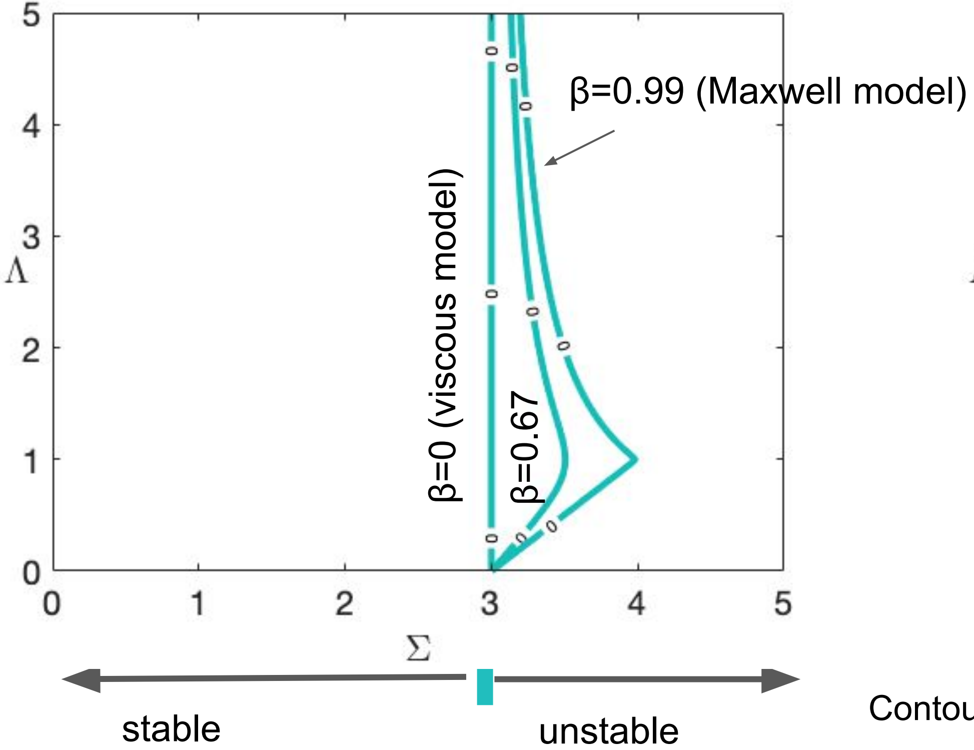
$$\omega\Lambda\hat{\sigma}_{B_y} + \hat{\sigma}_{B_y} + \omega\beta(\hat{\theta}_1 + \hat{\theta}_2) = 0$$

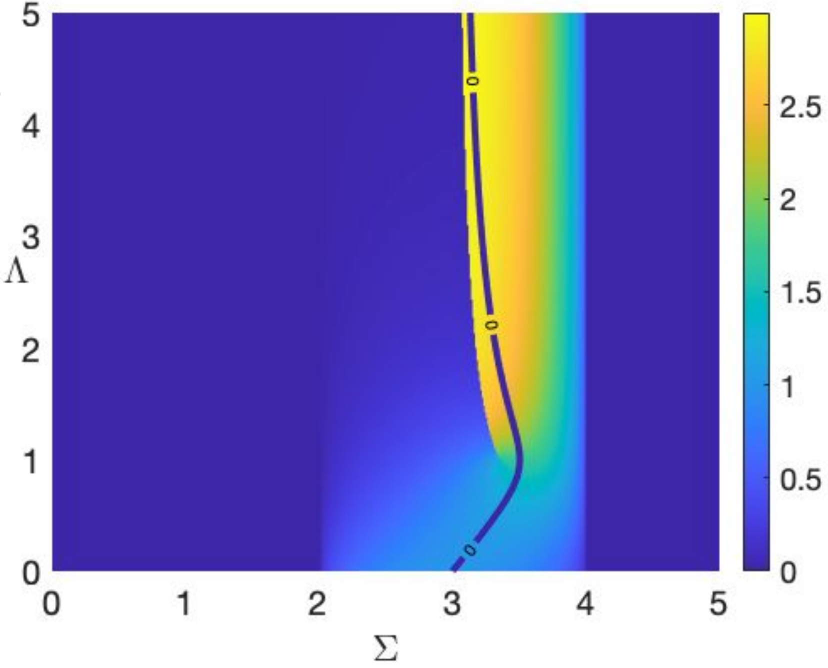
Separating the equations by $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\sigma}_{A_y}$, and $\hat{\sigma}_{B_y}$ yielded the matrix system

$$\begin{bmatrix} (\Sigma - 2\omega(1-\beta) - 2) & (-\Sigma - \omega(1-\beta) + 1) & 1 & 1 \\ -\omega(1-\beta) + 1 & -\omega(1-\beta) - 1 & 0 & 1 \\ \omega\beta & 0 & (\omega\Lambda + 1) & 0 \\ \omega\beta & \omega\beta & 0 & (\omega\Lambda + 1) \end{bmatrix} \begin{bmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\sigma}_{A_y} \\ \hat{\sigma}_{B_y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The solutions for ω are a bit too lengthy to use individually...

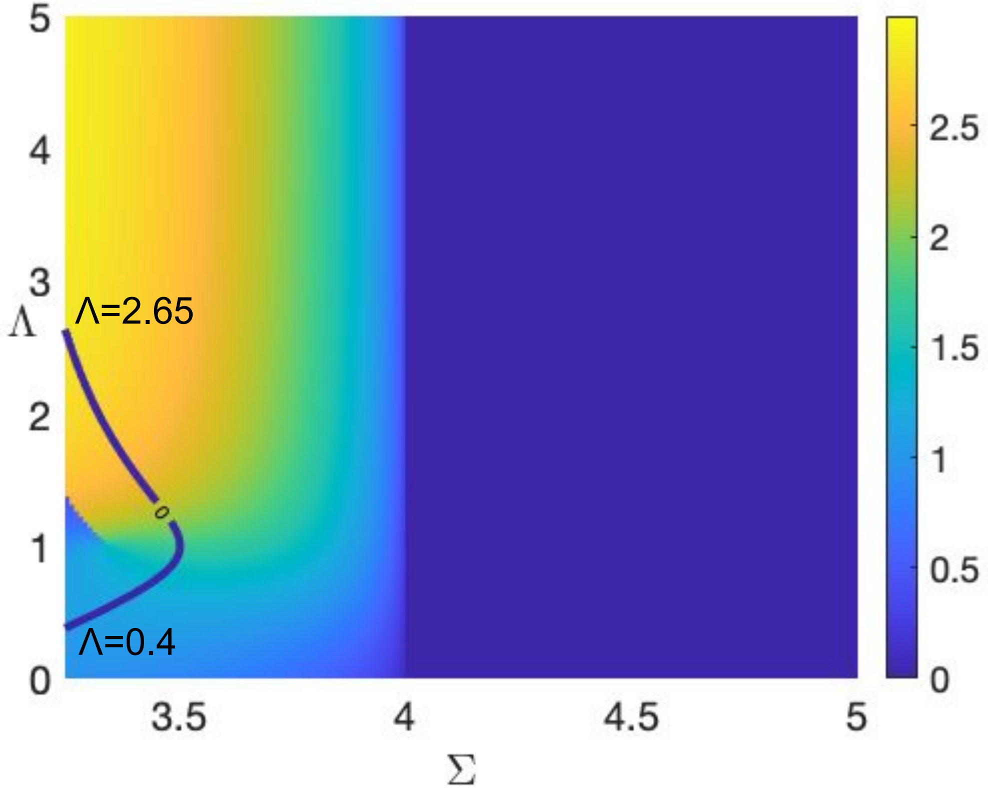
… So we solved it numerically!



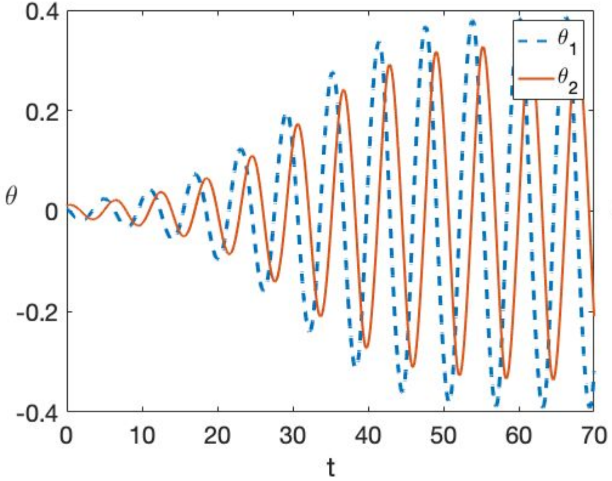


Contours of frequency overlaid with bifurcation curve for β=0.67

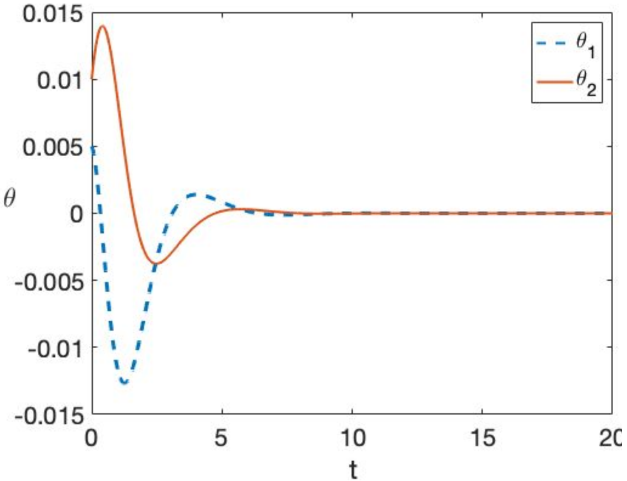
Two bifurcations along Σ=3.25 for β=0.67



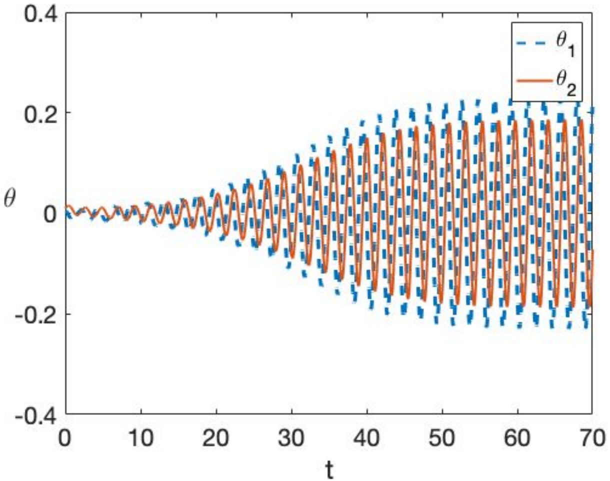
Two bifurcations along Σ=3.25



Λ=0.25

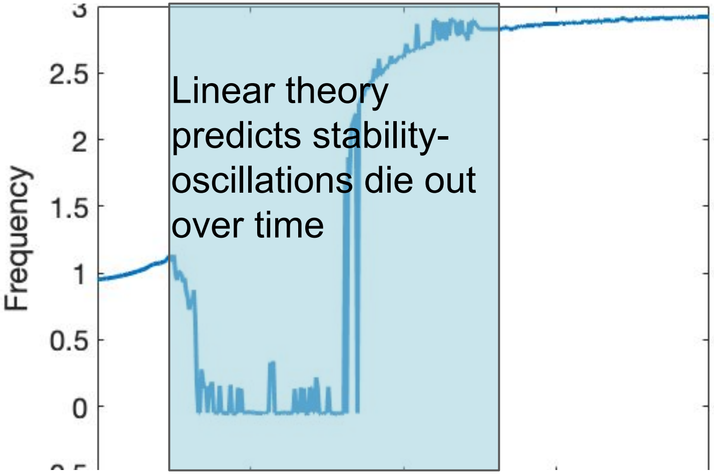


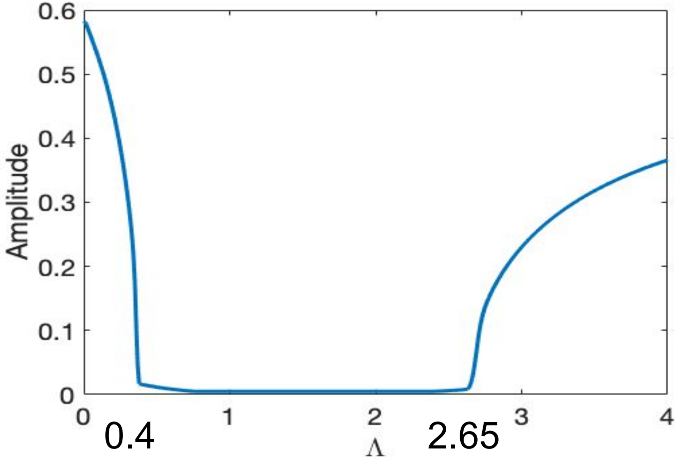
Λ=1

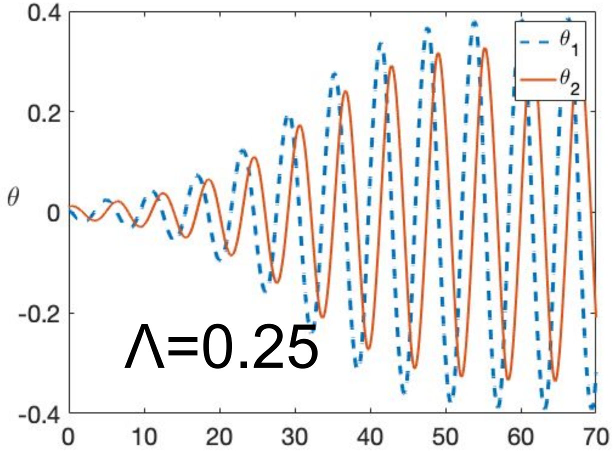


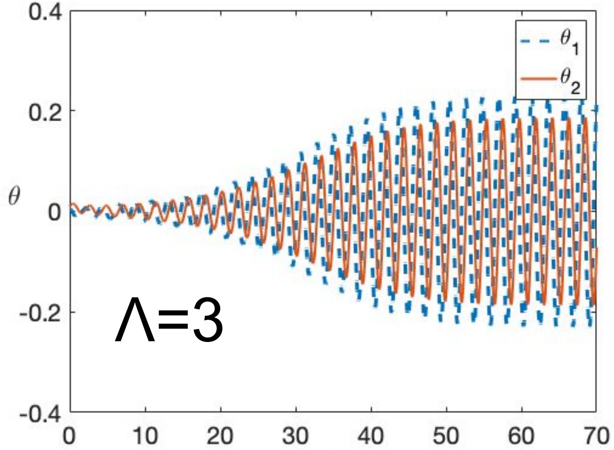
Λ=3

Frequency and amplitude change as we vary Λ

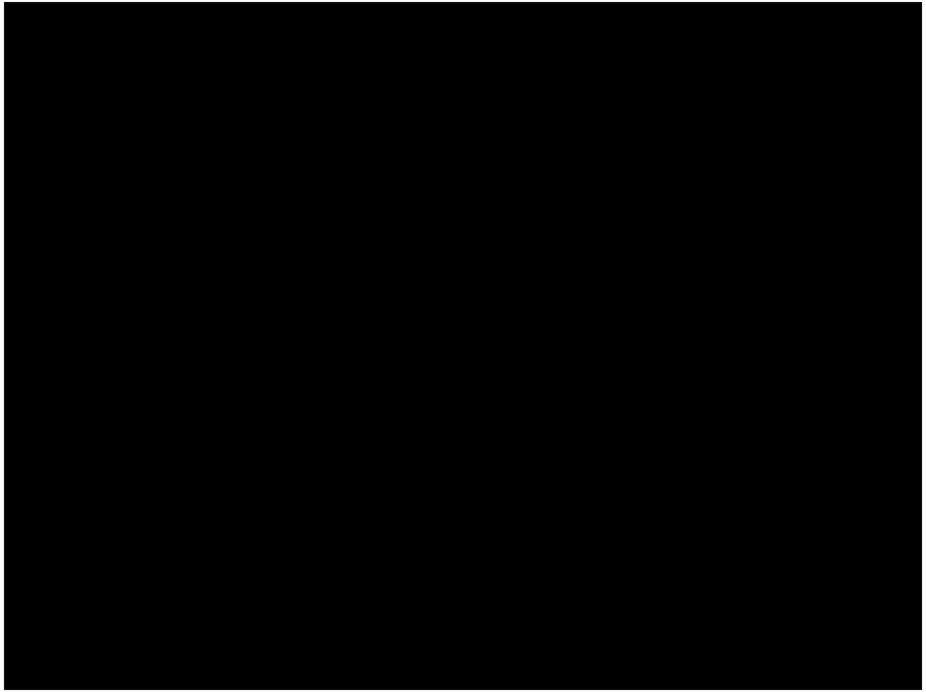




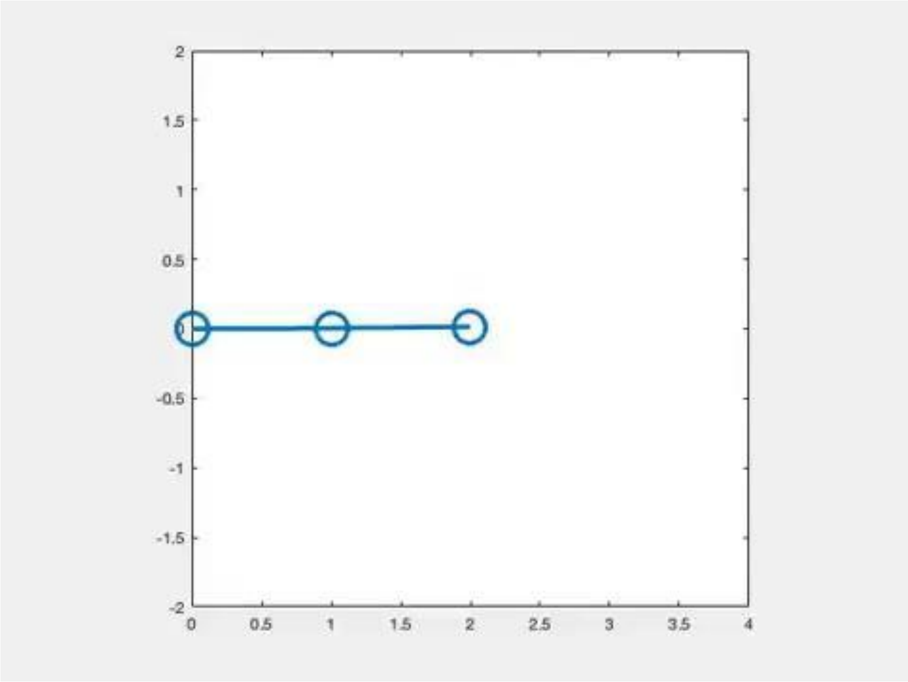




Frequency changes as we vary Λ



Λ=0.25



Λ=3

Conclusion-how does viscoelasticity affect flagellar motion?

- Maxwell model
  - Increased fluid viscoelasticity damps oscillations
- Oldroyd B Model
  - Two bifurcation points for viscoelastic fluids
  - Increasing viscoelasticity decreases oscillation amplitude, increases frequency

Thanks to...

Corey Beck, for collaborating with us.

Bob Guy, Becca Thomases, and Katie Link, for being wonderful mentors.

And all of you for making this such a fun and memorable summer!!!