Plane Partitions and Proctor's Miniscule Method: Explaining the $q = -1$ Phenomenon

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Abstract

This report introduces some concepts in Lie group representation theory with reference to their applications in studying plane partitions, following the treatments given by Kuperberg [3] and Proctor [8].

1 Introduction

Plane partitions are combinatorial objects that arise naturally as generalizations of integer partitions. Given an integer partition P with parts P_i (for $1 \leq i \leq k$) of some integer n, P can be represented visually as a chain of k different $1 \times P_i$ rectangles arranged in weakly decreasing height order (see Figure 1). Given a rectangle of size $a \times b$, the number of integer partitions that fit inside that rectangle when represented as described is $\binom{a+b}{a}$. Analogously, a *plane partition* in an $a \times b \times c$ box is a collection of unit cubes in the box stacked such that the heights of the stacks weakly decrease along both rows and columns, starting from the origin (see Figure 2).

In the early 1900s, Percy MacMahon discovered that the number of plane partitions in an $a \times b \times c$ box is

$$P(a,b,c) = \frac{H(a+b+c)H(a)H(b)H(c)}{H(a+b)H(a+c)H(b+c)},$$

where $H(n) = 1!2!...(n-1)!$ is the hyperfactorial function [6, pg.492-494]. He also defined two symmetry operations on plane partitions: transposition (τ) and rotation (ρ). *Transposition* is the

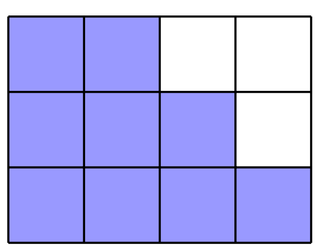


Figure 1: A partition of 9 with parts $3, 3, 2, 1$

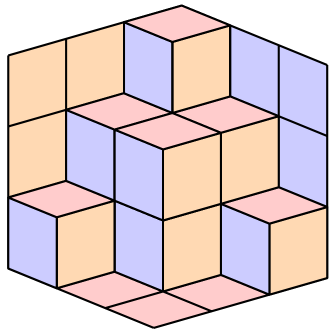


Figure 2: A plane partition

operation that maps each plane partition to the plane partition obtained by swapping the x and y axes (see Figure 3). *Rotation* maps each plane partition to the one obtained by a cyclic permutation of the axes (see Figure 4). Later, Mills *et al* defined a third symmetry operation, complementation (κ) [7]. Given a plane partition P in an $a \times b \times c$ box, its *complement* is the unique plane partition that, when rotated appropriately, uses all the cubes in the box that P does not (see Figure 6); *complementation* (κ) is the operation that maps each plane partition to its complement. All three classes together can be viewed as generating D_6, the group of symmetries of a regular hexagon. There are 10 conjugacy classes of subgroups of D_6, inducing 10 symmetry classes of plane partitions invariant under the various subgroups. Each of these symmetry classes has been enumerated separately (see Table 1)[1], and there are methods for enumerating several classes taken together, but there is no known single method for enumerating all of the classes taken together.

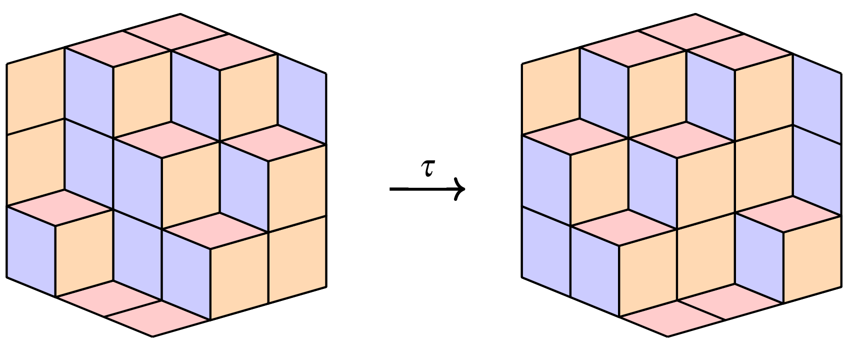


Figure 3: An example of transposition

There exist q-analogues for the formulas in Table 1 for the four symmetry classes which do not involve κ [5]. In particular,

$$P_q(a,b,c) = \frac{H(a+b+c)_q H(a)_q H(b)_q H(c)_q}{H(a+b)_q H(a+c)_q H(b+c)_q}, \tag{1}$$

where $H(n)_q = [1]_q![2]_q!\ldots[n-1]_q!$.[2]

[1] $H_k(n) = (n-k)!(n-2k)!(n-3k)!\ldots$ and $F_k(n) = n(n-k)(n-2k)\ldots$

[2] $[n]_q! = 1(1+q)(1+q+q^2)\ldots(1+q+\ldots+q^{n-1})$

Group	Enumeration
$\langle e\rangle$	$P(a,b,c)=\frac{H(a+b+c)H(a)H(b)H(c)}{H(a+b)H(a+c)H(b+c)}$
$\langle \tau\rangle$	$S(a,a,c)=\frac{H_2(2a+b+1)H(a)H_2(b)}{H_2(2a+1)H(a+b)}$
$\langle \rho\rangle$	$CS(a,a,a)=\frac{H_3(3a+2)H(a)}{H(2a)F_3(3a-2)}$
$\langle \tau,\rho\rangle$	$TS(a,a,a)=\frac{H_2(a)H_6(3a+5)}{H_2(2a+1)F_6(3a-2)}$
$\langle \kappa\rangle$	$SC(2a,2b,2c)=P(a,b,c)^2$
	$SC(2a,2b,2c+1)=P(a,b,c)P(a,b,c+1)$
	$SC(2a+1,2b+1,2c)=P(a+1,b,c)P(a,b+1,c)$
$\langle \kappa\tau\rangle$	$TC(a,a,2b)=\frac{H_2(2b+1)H_2(2b+2a)H(a)}{H(2b+a)H_2(2a)}$
$\langle \kappa,\tau\rangle$	$SSC(2a,2a,2b)=P(a,a,b)$
	$SSC(2a+1,2a+1,2b)=P(a,a+1,b)$
$\langle \kappa\tau,\rho\rangle$	$CSTC(2a,2a,2a)=\frac{F_3(3a-2)H_6(6a)H_2(2a)}{H_4(4a+1)H_4(4a)}$
$\langle \kappa,\rho\rangle$	$CSSC(2a,2a,2a)=\frac{H_3(3a+1)^2H(a)^2}{H(2a)^2}$
$\langle \kappa,\tau,\rho\rangle$	$TSSC(2a,2a,2a)=\frac{H_3(3a+1)H(a)}{H(2a)}$

Table 1: Plane Partition Enumeration Identities (from [10] and [9])

There are two different ways in which q-enumerations for these four symmetry classes can be defined. First, given a subgroup G of D_6, they can be defined by the sum

$$\sum_{\pi \text{ invariant under } G} q^{|\pi|},$$

where $|\pi|$ refers to the number of cubes in the plane partition π. Alternatively, they can be defined by the sum

$$\sum_{\pi \text{ invariant under } G} q^{|\pi/G|},$$

where $|\pi/G|$ refers to the number of orbits of G acting on π [10]. The first enumeration is q-round[3] for the subgroups $\langle e\rangle$, $\langle\rho\rangle$, and $\langle\tau\rangle$, but not for $\langle\tau,\rho\rangle$. The second enumeration is q-round for the subgroups $\langle\tau,\rho\rangle$ and $\langle\tau\rangle$.

[3] A number is round if it is the product of (relatively) small primes. If the numbers in a sequence are round, this is a good indication that the sequence has a nice underlying pattern. A polynomial in q is q-round if it is a product of cyclotomic polynomials.

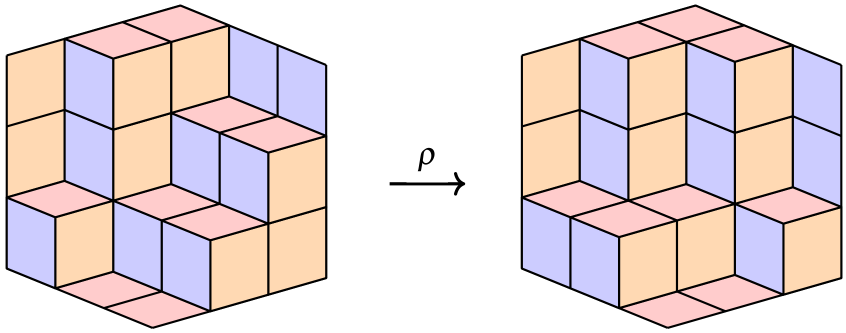


Figure 4: An example of rotation

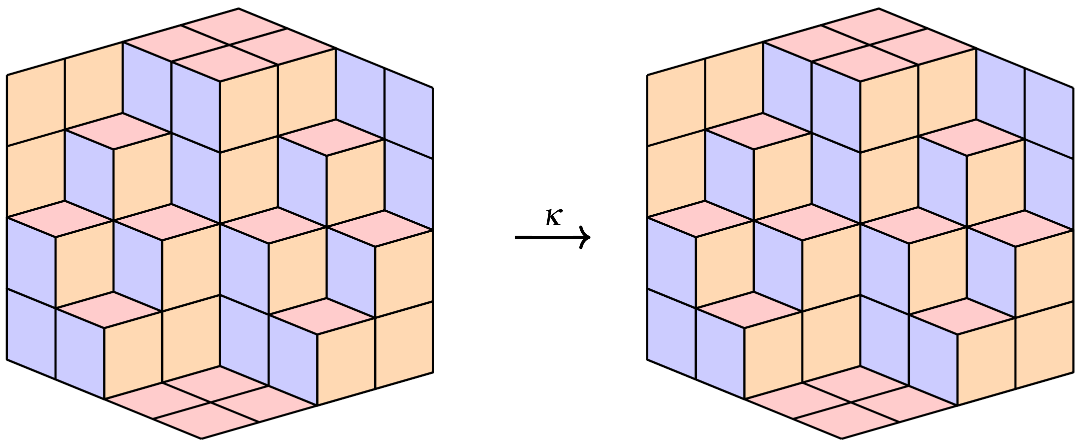


Figure 5: A complementation-invariant plane partition

These q-enumerations lead to some interesting enumeration identities. In particular, setting $q = -1$ in $P_q(2a,2b,2c)$ yields a formula identical to the formula for the number of self-complementary plane partitions in a $2a \times 2b \times 2c$ box, *i.e.*

$$SC(2a,2b,2c) = P_{-1}(2a,2b,2c).$$

Since a plane partition with k cubes has a weight of q^k in (1), this identity is equivalent to the statement that, in a $2a \times 2b \times 2c$ box, the number of plane partitions using an even number of cubes minus the number of plane partitions using an odd number of cubes is the same as the number of self-complementary plane partitions. In [11], Stembridge mentions a total of 6 similar $q = -1$ phenomena. One remarkable aspect of these $q = -1$ phenomena is that they work using either of the q-enumerations given above; the -1-enumerations are round in cases where the corresponding q-enumerations are not.

These $q = -1$ phenomena are rather surprising and were first shown by comparing already derived formulas. However, this method of proof does not constitute an entirely satisfactory explanation of the phenomenon; it seems like there should be some deeper explanation why, for example, the number of self-complementary plane partitions is the same as the evaluation of a particular polynomial at $q = -1$. The remainder of this report will introduce the representation theory of semisimple, complex Lie algebras and outline a particular connection of plane partitions with representation theory given by Kuperberg in [3] that yields a rather more satisfactory explanation of the $q = -1$ phenomenon.

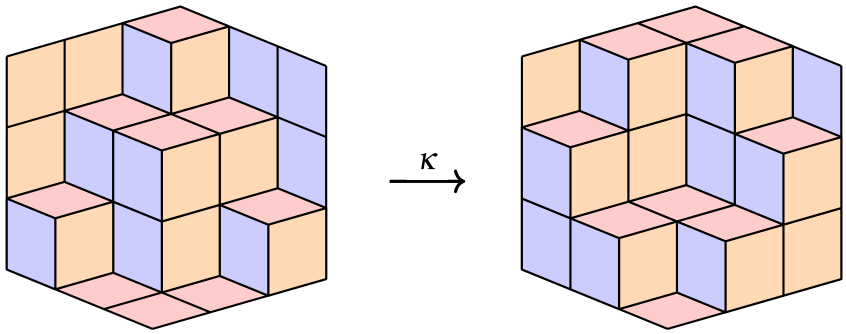


Figure 6: An example of complementation

2 Lie Groups and Lie Algebras

A *Lie group* G is a C^∞ manifold with C^∞ multiplication and inverse operations. An example is the complex unit circle S^1. Intuitively, it is a smooth manifold because locally it 'looks just like' $\mathbb{R}$; every point on the manifold has a neighbourhood homeomorphic to an open subset of $\mathbb{R}$. It is a group under complex multiplication, and both the multiplication and inverse maps can be shown to be continuous. Other important examples of Lie groups include the classical Lie groups $SO(n)$ (the special orthogonal group), $O(n)$ (the orthogonal group), $Sp(2n)$ (the symplectic group), $U(n)$ (the unitary group), $SU(n)$ (the special unitary group), $GL(n,\mathbb{C})$, and $SL(n,\mathbb{C})$.

Given a connected Lie group G, the corresponding *Lie algebra* L is the tangent space to G at the identity. The tangent space to G at 1 corresponds to all tangent vectors $A'(0)$ to smooth paths $A(t)$ in G, where $A(t)$ satisfies that $A(0) = 1$. For example, the tangent space to S^1 is the vertical tangent line to the circle passing through 1. Lie algebra elements can be thought of as "infinitesimal generators" of a Lie group G, in the sense that $\exp(L)$ often contains all, or almost all, elements of G [12, pg.45]. In the abstract, a Lie algebra is a vector space equipped with a bilinear map called the Lie bracket $[\cdot,\cdot] : L \times L \to L$ satisfying the Jacobi identity

$$\forall x,y,z \in L : [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$$

and satisfying that $\forall x \in L : [x,x] = 0$. This implies that, in particular, Lie algebras are generally not associative. In the context of Lie groups, the Lie bracket of a Lie algebra arises naturally from the group commutator in the corresponding Lie group, and is a way of capturing the non-commutative aspects of the group [12]. The algebras corresponding to the classical Lie groups can all be formed from the Lie algebra $\mathfrak{gl}(n,\mathbb{C})$ of all $n \times n$ matrices over $\mathbb{C}$ with Lie bracket defined by the matrix commutator $[x,y] = xy - yx$; for example, $\mathfrak{sl}(n,\mathbb{C})$, the Lie algebra corresponding to $SL(n,\mathbb{C})$, is the subspace of $\mathfrak{gl}(n,\mathbb{C})$ consisting of matrices M with $tr(M) = 0$.

Given a group G, a *representation* of G on a vector space V is a homomorphism $\phi : G \to GL(V)$; by abuse of notation, V is generally referred to as a representation of G when ϕ is clear. Lie algebra representations are defined similarly. Of particular interest are *irreducible representations*, those representations which do not have any nontrivial proper subrepresentations. A representation can be thought of as a way of describing what a group G 'looks like' by describing what it does to a particular vector space; for example, one representation of the group S_3 is as a subgroup of $GL(2,\mathbb{C})$ consisting of rotation and reflection matrices. In the context of Lie algebras, one particularly important example of a representation is the adjoint representation, the representation given by

$$ad : L \to gl(L)\ ,\ (ad(x))(y) = [x,y]$$

for L a Lie algebra.

There exists a nice relationship between representations of Lie groups and representations of Lie algebras. If G is simply connected, then every representation of G corresponds uniquely to a representation of its Lie algebra L, and vice versa [2, 5.7]. In particular, this correspondence holds in the case of $SL(n,\mathbb{C})$ and $\mathfrak{sl}(n,\mathbb{C})$. From here on, only the Lie algebra representation theory will be considered, using the fact that, for the purposes of explaining the $q=-1$ phenomenon, $SL(n,\mathbb{C})$ is a particular Lie group of interest. What follows is a brief introduction to some of the ideas and terms important to the study of the representations of complex, semisimple Lie algebras, of which $\mathfrak{sl}(n,\mathbb{C})$ is one; for a more comprehensive and detailed treatment, see [1] or [2].

Let L be a Lie algebra, V a representation of L, and H an abelian subalgebra of L. Then a *weight* is a linear function $\lambda : H \to \mathbb{C}$, and a *weight space* V_λ corresponding to a weight λ is defined by

$$V_\lambda = \{v \in V : av = \lambda(a)v \text{ for all } a \in H\}.$$

Intuitively, a weight is a kind of generalized eigenvalue and V_λ is the corresponding eigenspace.

Given a complex, semisimple Lie algebra L, a *Cartan subalgebra* H is defined to be a maximal abelian diagonalizable subalgebra of L. Since a Cartan subalgebra H is abelian and consists of semisimple elements, by the Jordan decomposition[4], each element can be diagonalized. If linear maps commute, they can be simultaneously diagonalized, and so, since the adjoint representation preserves diagonalizability, H acts diagonalizably on L in the adjoint representation. So, L can be decomposed into a direct sum of weight spaces for the adjoint action of H [1, pg.92]. In other words, for $\alpha \in H^\star$, letting

$$L_\alpha = \{x \in L : [h,x] = \alpha(h)x\ \forall h \in H\}$$

and

$$\Phi = \{\alpha \in H^\star : L_\alpha \neq 0\},$$

$$L = H \oplus \left(\bigoplus_{\alpha\in\Phi} L_\alpha \right).$$

The elements of Φ are called *roots*. $H^\star$ is an inner product space under the inner product $(\alpha,\beta) = \kappa(t_\alpha,t_\beta)$, where κ is the Killing form $\kappa(x,y) = tr(ad(x) \circ ad(y))$ and t_α is the unique element of H such that $\kappa(t_\alpha,k) = \alpha(k)$ for all $k \in H$. Such a t_α is guaranteed to exist since κ is non-degenerate and the map sending $h \to \theta_h$, where $\theta_h : H \to H^\star$ is defined by $\theta_h(k) = \kappa(h,k)$, is an isomorphism between H and $H^\star$. Since $H^\star$ is an inner product space, it makes sense to speak of the roots of a Lie algebra as vectors and to speak of the angle between two roots.

A *base* for a root system is a subset B of the root system such that B is a basis for $H^\star$ and every element of the root system is a linear combination of elements of B where the coefficients in the linear combination all have the same sign. To every root can be associated a reflection s_α, which is the reflection in the hyperplane normal to α. The group generated by the reflections corresponding

[4]The Jordan decomposition states that each $x \in L$ can be expressed as $x = d+n$, where ad_d is diagonalizable, ad_n is nilpotent, and $[d,n] = 0$. If an element is x semisimple, it has $n = 0$ in its Jordan decomposition, and representations of a semisimple Lie algebra preserve the Jordan decomposition (*i.e.*$\phi(x) = \phi(d)+\phi(n)$).

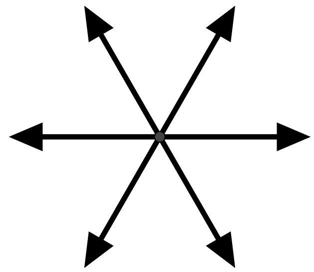


Figure 7: The root system corresponding to $\mathfrak{sl}(3,\mathbb{C})$

to simple roots is called the *Weyl group* of the root system; the Weyl group is always finite, and becomes important in the representation theory of Lie algebras. In particular, the set of weights of a representation is always invariant under the action of the Weyl group.

The roots of a Lie algebra play an important role in its representation theory. A partial order on the set of weights of a representation, sometimes called the standard partial order, can be defined by setting $\mu \succeq \nu$ if $\mu - \nu$ is a positive linear combination of positive roots. The irreducible representations of a complex semisimple Lie algebra can be classified by their unique highest weights; in fact, there is a bijection between fundamental dominant weights and irreducible representations V^{λ} of a complex semisimple Lie algebra L, where, for Π a base for the root system of L, a *fundamental dominant weight* λ satisfies that $\langle \lambda, \alpha \rangle \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in \Pi$. This classification allows for the construction in the next section of an irreducible representation with basis elements precisely corresponding to plane partitions.

3 Proctor's Miniscule Method

This section closely follows the treatment given in [3]. The idea is to build an irreducible representation of $SL(a+b,\mathbb{C})$ with basis elements corresponding to plane partitions, then consider actions of $SL(a+b,\mathbb{C})$ on that space.

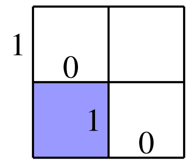
A plane partition in an $a \times b \times c$ box can be thought of as a chain of c rectangle partitions of dimension $a \times b$. Any rectangle partition P in an $a \times b$ box is uniquely determined by its boundary, which consists of a path from the upper left corner to the lower right corner of the rectangle along $a+b$ line segments, where each step goes either down or to the right (see Figure 8). Define a function $b_P(n) : \{1,...,a+b\} \to \{0,1\}$ by

$$b_P(n) = \begin{cases} 1, & \text{if the } n\text{th step goes down} \\ 0, & \text{otherwise} \end{cases}.$$

Note that there are a values of n such that $b_P(n) = 1$.

This structure can be encoded in terms of vector spaces as follows. Let V be a complex vector space with basis $x_1,...,x_{a+b}$, and consider the ath exterior power of V, $\Lambda^a(V)$, consisting of alternating forms over V with degree a. The set

$$B = \{x_{i_1} \wedge ... \wedge x_{i_a} : 1 \leq i_1 < i_2 < ... < i_a \leq a+b\}$$



$$b_P(n) = \begin{cases} 1, & \text{n=1 or n=3} \\ 0, & \text{n=2 or n=4} \end{cases}$$

Figure 8: A rectangle partition P with values of $b_P(n)$ labelled

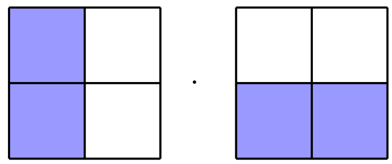


Figure 9: A chain of rectangle partitions that is not a plane partition

is a basis for this space. This basis can be equivalently expressed as

$$B = \left\{ x_P = \bigwedge_{i=1}^{a+b} x_i^{b_P(i)} : P \text{ is an } a \times b \text{ rectangle partition} \right\}.$$

This space is a representation of $SL(n, \mathbb{C})$.

Now, consider the space $S^c(\Lambda^a(V))$ of symmetric polynomials of degree c over $\Lambda^a(V)$. Given a basis W for $\Lambda^a(V)$, the set X_W of monomials in W of degree c is a basis for $S^c(\Lambda^a(V))$. Using the basis B for $\Lambda^a(V)$ defined above, every plane partition in an $a \times b \times c$ box can be identified with a basis element of $S^c(\Lambda^a(V))$ using the basis X_B; namely, given a plane partition P that can be expressed as a chain of rectangle partitions $P_1, \ldots, P_c$, P can be identified with the basis element

$$\prod_{i=1}^{c} x_{P_i}.$$

This space is also a representation of $SL(a+b, \mathbb{C})$; however, it is not the representation that we are looking for, as it contains basis elements that do not correspond to plane partitions. For example, the representation $S^2(\Lambda^2(V))$ of $SL(4, \mathbb{C})$ (where V is 4-dimensional) contains an element corresponding to the chain of rectangle partitions pictured in Figure 9, a chain which cannot correspond to a plane partition.

This is where Proctor's method comes in to play. His paper examines the weight lattices corresponding to irreducible minuscule representations of simple Lie algebras, where a *minuscule representation* is an irreducible representation V^λ for which every every weight w of V^λ is of the form $w\lambda$ for some w in the corresponding Weyl group W [8, section 4]. The weights of such a representation V^λ form a lattice under the standard partial order. Referencing work by Seshadri, Proctor states that there is a basis for $V^{m\lambda}$ indexed by m-multichains in the lattice of weights of V^λ, where an m-multichain in a lattice is a set $x_1 \leq x_2 \leq \ldots \leq x_m$ of elements that appear in that lattice.

$\Lambda^a V$ happens to be the irreducible miniscule representation of $SL(a+b, \mathbb{C})$ corresponding to the ath fundamental weight λ_a, and the basis B described above happens to consist of weight vectors.

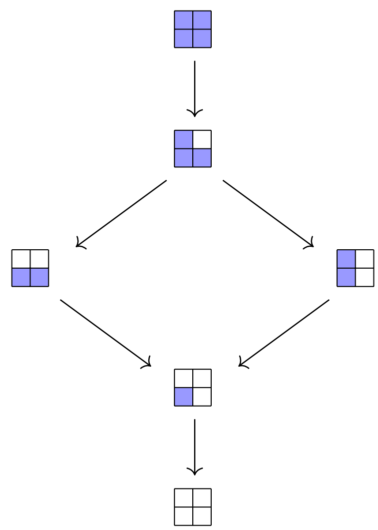


Figure 10: Rectangle partitions ordered by inclusion

The lattice of basis elements of $\Lambda^a(V)$ is the same as the lattice of rectangle partitions ordered by inclusion (Figure 10), and c-multichains in this lattice correspond exactly to plane partitions and thus to basis elements of $S^c(\Lambda^a(V))$. Note that no multichain that does not correspond to a plane partition can occur, since multichains are defined with respect to the partial order and thus situations like that illustrated in Figure 9 are completely avoided, fixing our problem with $S^c(\Lambda^a(V))$. So, there is a subrepresentation $V(c\lambda_a)$ in $S^c(\Lambda^a(V))$ with basis corresponding exactly to plane partitions; further, there is an equivariant projection

$$\pi : S^c(\Lambda^a(V)) \to V(c\lambda_a)$$

(*i.e.* π commutes with the group action of $SL(a+b,\mathbb{C})$ on $S^c(\Lambda^a(V))$).

This sequence of representations allows various plane partition enumerations to be easily computed. In fact, the number of plane partitions in an $a\times b\times c$ box is immediate from this construction. The Weyl dimension formula gives the dimension of an irreducible representation of a complex, semisimple Lie algebra in terms of its weights; namely, if L is a Lie algebra, R^+ is the set of positive roots of the root system corresponding to L,

$$\delta = \frac{1}{2}\sum_{\alpha\in R^+}\alpha,$$

and V_μ is the irreducible representation with highest weight μ, then the Weyl dimension formula states that

$$\dim(V_\mu) = \frac{\prod_{\alpha\in R^+}\langle\alpha,\mu+\delta\rangle}{\prod_{\alpha\in R^+}\langle\alpha,\delta\rangle}.$$

See [2, Section 10.5] for more details. In the case of the representation $V(c\lambda_a)$ constructed above, this can be computed to be equivalent to $P(a,b,c)$.

For other enumerations, the idea is to compute the trace of elements of $SL(a+b,\mathbb{C})$ in their actions on $V(c\lambda_a)$ by computing their eigenvalues, pulling the computation through the constructed sequence of representations. Permutation matrices will be of particular use, since the trace of a permutation matrix is the number of elements it fixes. For example, the trace of the identity matrix $I \in SL(a+b,\mathbb{C})$ in its action on $V(c\lambda_a)$ is just the dimension of $V(c\lambda_a)$, and the trace of

$$D_q = q^{(1-a-b)/2}\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & q & 0 & \dots & 0 \\ 0 & 0 & q^2 & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & q^{a+b-1} \end{bmatrix}$$

can be shown to be $P_q(a,b,c)$ using the q-analogue of the Weyl dimension formula.

Now, returning to the example of the $q=-1$ phenomenon in Section 1 (namely, $SC(2a,2b,2c) = P_{-1}(2a,2b,2c)$), the idea is to construct an element K of $SL(a+b,\mathbb{C})$ with trace equal to $SC(2a,2b,2c)$ and relate that element to the matrix D_q given above for $q=-1$. The key insight here is that complementing a plane partition is equivalent to reversing the binary sequences in each of its constituent rectangle partitions; for example, see Figure 11. The matrix that acts on rectangle partitions to reverse their binary sequences is

$$K = i^{a+b-1}\begin{bmatrix} 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 1 & 0 & 0 \\ . & . & . & \dots & . \\ 1 & \dots & 0 & & 0 \end{bmatrix},$$

where the i^{a+b-1} term ensures that K has determinant 1 and is thus an element of $SL(a+b,\mathbb{C})$. By construction, K commutes with π, and thus it also complements plane partitions. K also happens to be conjugate to D_{-1}; in other words, the action of D_{-1} is the same as the action of K, just in a different basis. Conjugate matrices have the same trace, proving that $SC(2a,2b,2c) = P_{-1}(2a,2b,2c)$, and giving a more satisfactory explanation of why the $q=-1$ phenomenon occurs.

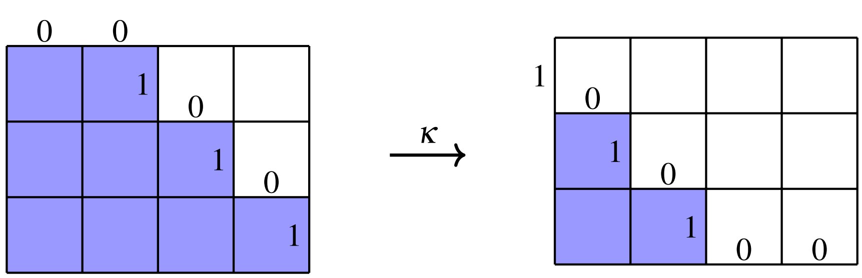


Figure 11: Rectangle partition complementation

4 Conclusion

This connection of plane partitions with representation theory, in addition to admitting a model of complementation and explaining certain $q = -1$ phenomena, admits a model of transposition. It thus gives a unified description of five symmetry classes of plane partitions, namely P, S, SC, TC, and (at least conjecturally) SSC from Table 1. There is another, quite different connection with representation theory given in [4] that explains four symmetry classes, namely P, CS, TC, and $CSTC$. However, there is not currently a representation theory model that explains all ten symmetry classes; this would be an interesting direction for future work.

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