The Markov Graph: A New Approach and an Exception

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Outline

Introduction

Markov Equation

$$x_1^2 + x_2^2 + x_3^2 - 3x_1x_2x_3 = 0$$

- Markov Triple: integer solution

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  - DRIVING PROJECT GOAL: Is $\hat{G}_p$ connected?

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- Markov Triple: integer solution
- Reduction: modulo a prime p
- $\hat{G}_p$: a graph — vertices are Markov triples (mod p)
  - DRIVING PROJECT GOAL: Is $\hat{G}_p$ connected?
- Bourgain, Gamburd, and Sarnak (BGS): one approach to connectivity — growing a certain known connected component ("the cage")

Project Background: Constructing $\hat{G}_p$

Functions Preserving Markov Triples

Involutions:

$$R_1(x_1, x_2, x_3) = (3x_2x_3 - x_1, x_2, x_3)$$
$$R_2(x_1, x_2, x_3) = (x_1, 3x_1x_3 - x_2, x_3)$$
$$R_3(x_1, x_2, x_3) = (x_1, x_2, 3x_1x_2 - x_3)$$

Rotations:

$$\mathrm{rot}_{x_1}(x_1, x_2, x_3) = (x_1, x_3, 3x_1x_3 - x_2)$$
$$\mathrm{rot}_{x_2}(x_1, x_2, x_3) = (3x_2x_1 - x_3, x_2, x_1)$$
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- $\hat{G}_p$ edges: ROTATIONS

Project Background: Constructing $\hat{G}_p$

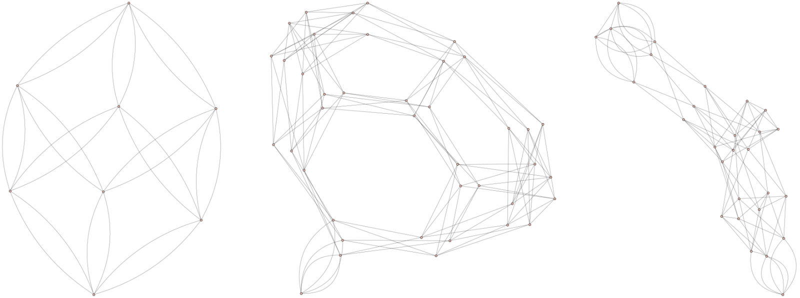
Example

Let $p = 13$. Rotation:

- Markov Triple: $(3, 0, 11)$
- $3^2 + 0^2 + 11^2 - 3(3)(0)(11) = 130 \equiv 0 \mod 13$
- $\text{rot}_3(3, 0, 11) = (3, 11, 3(3)(11) - 0) \equiv (3, 11, 8)$
- $3^2+11^2+8^2-3(3)(11)(8) = 194-792 = -598 = -46(13) \equiv 0 \mod 13$
- $(3, 11, 8)$ is a Markov Triple

Project Background: Constructing $\hat{G}_p$

$\hat{G}_p$ for $p = 3, 5,$ and 7



Project Background: Constructing $\hat{G}_p$

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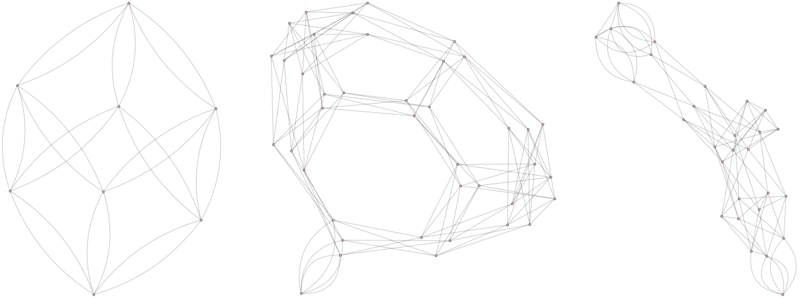

Question:

But why rotations?

Project Background: Introducing Order

Rotations as Matrices

- $\text{rot}_{x_1}\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix}\begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$, where $x = 3x_1$
- $\text{rot}_{x_1}(x_1, x_2, x_3) = (x_1, x_3, 3x_1x_3 - x_2)$

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- $\text{rot}_{x_1}(x_1, x_2, x_3) = (x_1, x_3, 3x_1x_3 - x_2)$
- $$\begin{pmatrix} 0 & 1 \\ -1 & x \end{pmatrix} \in \text{SL}_2(\mathbb{F}_p)$$
- The **order** of some $x \in \{0, ..., p-1\}$ is the order of the rotation matrix of x in the group $\text{SL}_2(\mathbb{F}_p)$. (This is *finite*.)
  - **Orbit** of a rotation on a Markov triple: All triples obtained by iteratively applying the rotation to some Markov triple.
  - An x of "maximum" order is **maximal**.
  - Order of a *triple*: largest order of its components

Project Background: The Cage

The **Cage**: all MAXIMAL Markov triples

Theorem (Bourgain-Gamburd-Sarnak)

The cage is connected in $\hat{G}_p$.

New Approach with Parabolic Cycles

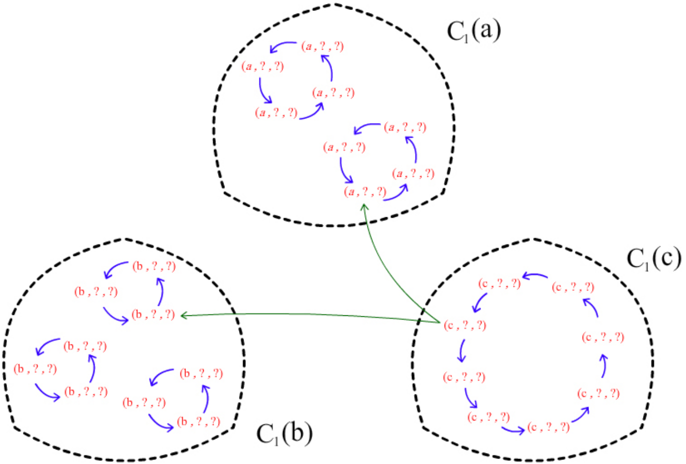
Organizing Markov Triples

- $C_1(a)$: All Markov triples with first coordinate equal to a (BGS).
- E.g. for $p = 7$,

$$C_1(2) = \{(2,1,1),(2,1,5),(2,2,6),(2,5,1),(2,6,2),(2,6,6)\}$$

- Add in rotations...

New Approach with Parabolic Cycles



- Arrows: rotations (rot_a, "jumping" rotations)

New Approach with Parabolic Cycles

Parabolic Elements

- Elements in $\{0, \ldots, p-1\}$ may be classified into one of three types: parabolic, hyperbolic, and elliptic (BGS)
- Parabolic elements exist iff $p \equiv 1 \mod 4$
- Only two parabolic elements: $x \equiv \pm 2 \mod p$
- Parabolic elements are *always* maximal

New Approach with Parabolic Cycles

Previously Known

Choose any $t \in \{0, \dots, p-1\}$. There exists a Markov triple containing t which has another coordinate equal to a parabolic element.

- Why? Recall $x \equiv \pm 2 \mod p \implies x_1 = \pm\frac{2}{3}$.
- BGS: For any $t \in \mathbb{F}_p$ (with $i^2 \equiv -1 \mod p$),

$$
\begin{aligned}
C_1(\tfrac{2}{3}) &= (\tfrac{2}{3}, t, t \pm \tfrac{2i}{3}) \\
C_1(-\tfrac{2}{3}) &= (-\tfrac{2}{3}, t, -t \pm \tfrac{2i}{3})
\end{aligned}
$$

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$$C_1(\frac{2}{3}) = (\frac{2}{3}, t, t \pm \frac{2i}{3})$$
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Question: How can we restate the above statement in terms of $C_1(a)$'s?

New Approach with Parabolic Cycles

Restated

$C_1(a)$ **contains a parabolic triple for every** $a \in \{0, ..., p-1\}$ **if** $p \equiv 1 \mod 4$**.**

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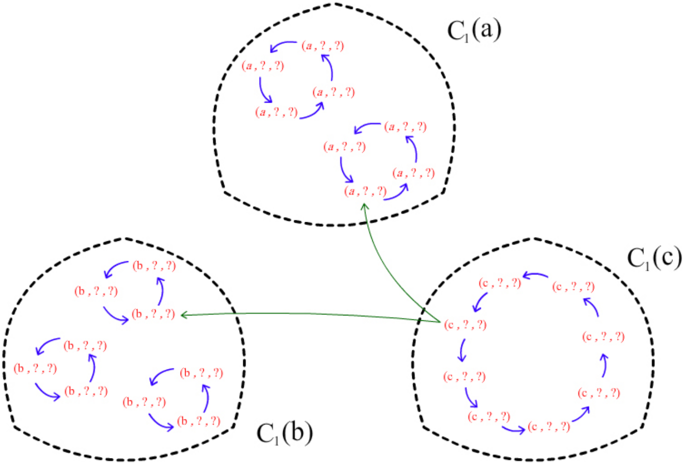
$C_1(a)$ contains a parabolic triple for every $a \in \{0, ..., p-1\}$ if $p \equiv 1 \mod 4$.

Theorem

For every $a \in \{0, ..., p-1\}$, $C_1(a)$ contains either 6 or 8 parabolic triples if $p \equiv 1 \mod 4$.

- Recall: Every parabolic triple is maximal and connected to the cage (which is connected).

New Approach with Parabolic Cycles



Blue rot_a orbits containing parabolic triples are **parabolic cycles**.

New Approach with Parabolic Cycles

Prime Modulus	Parabolic Cycles/Orbits	Trpls. in P. Cyc./Total
5	100%	100%
13	85%	91.667%
17	82.759%	89.583%
29	62.5%	77.857%
37	70.513%	86.111%
41	61.345%	78.429%
53	61.364%	80.342%
61	58.333%	82%
73	55.385%	81.404%
89	45.161%	72.446%
97	47.923%	77.220%
101	49.102%	75.369%

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- Visualize structure of $\hat{G}_p$

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Approach Advantages

- Visualize structure of $\hat{G}_p$
- Parabolic elements easy to find ($\pm 2 \mod p$) $\implies$ parabolic cycles easy to find
- Mathematical insights to assess connectedness:
  - Order of $a \in \mathbb{F}_p$ indicates connectivity for an entire class of Markov triples (in $C_1(a)$)
  - Distributions of orders in $\mathbb{F}_p$ indicates connectivity for $\hat{G}_p$

A Found Exception

Examining orbital structures in $C_1(a)$'s $\implies$ seeing MANY examples.
One was especially interesting.

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Theorem
Suppose we exclude the trivial Markov solution $(0, 0, 0)$. Then,

1. if $p \equiv 1 \mod 4$, $|C_1(0)| = 2(p-1)$, and
2. if $p \equiv 3 \mod 4$, $|C_1(0)| = 0$.

A Found Exception

An Exception?

- A fundamental previous result by BGS implied
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- The proof was merely sketched...
- But for all $a \neq 0$, the BGS result has been experimentally accurate.
- "Wait...so what's *actually* going on?"

A Found Exception

Proof Sketch

When $x_1 = 0$, the Markov equation simplifies

$$x_1^2 + x_2^2 + x_3^2 - 3x_1x_2x_3 \equiv 0 \mod p$$
$$\implies x_2^2 \equiv -x_3^2 \mod p$$

The number of solutions, and size of $C_1(a)$, depends upon $(\frac{-1}{p})$. This is equivalent to asking $p \mod 4$. If $p \equiv 1 \mod 4$, there are $p - 1 = |\mathbb{F}_p^*|$ choices for x_3 and 2 choices for x_2, namely $\pm x_2$. If $p \equiv 3 \mod 4$, there are no solutions.

Thank you and Questions

Thank you to Elena and Daniel for invaluable guidance, support, and patience.
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Thank you to you for listening.