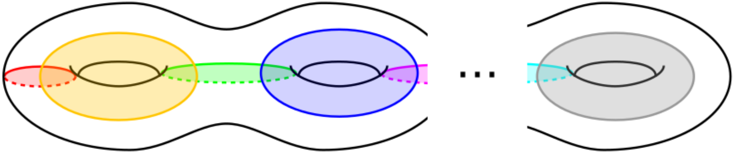
Our UC Davis REU Experience

Arboreal Lagrangian skeleta for 4-manifolds



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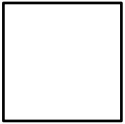
Section 1

Smooth handle decompositions

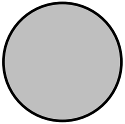
Smooth manifolds

A smooth n-manifold M is a topological space covered by charts (U, φ) such that $\varphi : U \to \mathbb{R}^n$ is a homeomorphism and, for any two charts (U, φ) and (V, ψ), the map $\varphi \circ \psi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism. For a smooth n-manifold with boundary, we can replace $\mathbb{R}^n$ with $\mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$.

not a smooth manifold

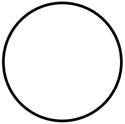


smooth 2-manifold with boundary



$$D^2 = \{(x, y) : x^2 + y^2 \leq 1\}$$

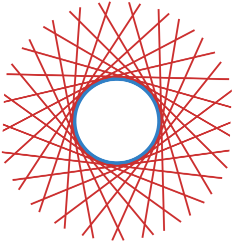
smooth 1-manifold without boundary

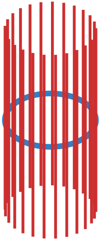


$$\partial D^2 = S^1 = \{(x, y) : x^2 + y^2 = 1\}$$

At each point $x \in M$, there is an n-dimensional real vector space T_xM consisting of the vectors tangent to M at x. The tangent bundle TM is the smooth $2n$-manifold

$$TM = \coprod T_xM = \{(x, v) : x \in M, v \in T_xM\}.$$

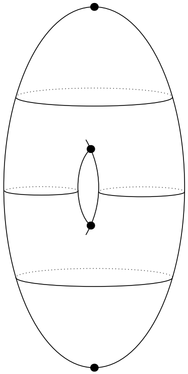




The dual of the tangent bundle TM is the cotangent bundle

$$T^*M = \coprod T_x^*M = \{(x, \varphi) : x \in M, \varphi : T_xM \to \mathbb{R} \text{ is linear}\}.$$

papa's donuteria



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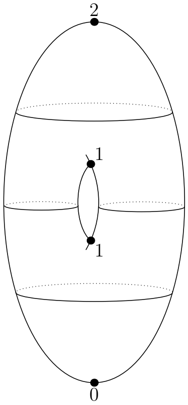
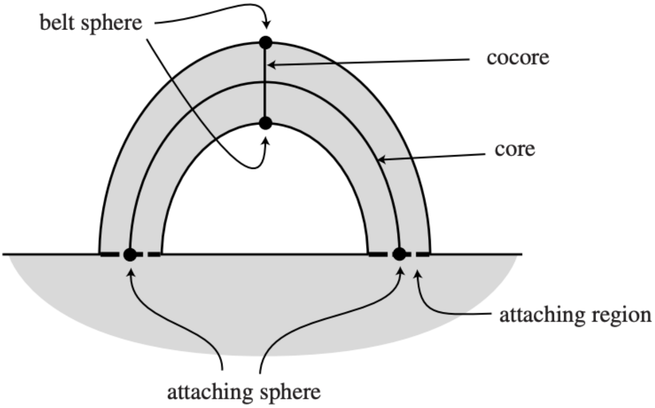
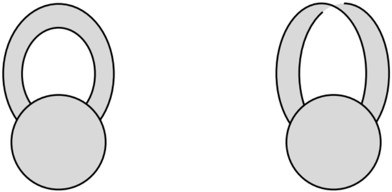


Handles and company

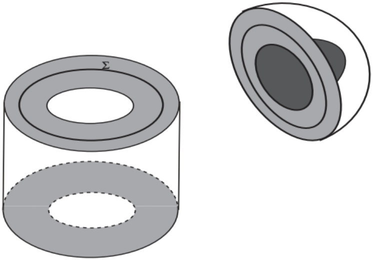
For $0 \leq k \leq n$, an n-dimensional k-handle is a copy of $D^k \times D^{n-k}$ with an attaching embedding $\varphi : \partial D^k \times D^{n-k} \to \partial M$.



Examples of handles



Two ways to attach a 1-handle to D^2, depending on orientability

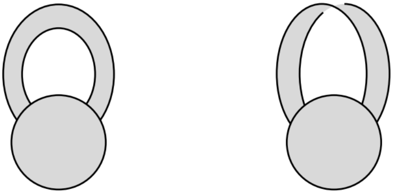


Attaching a 3-dimensional 2-handle along Σ

Diffeomorphism type of a handle attachment

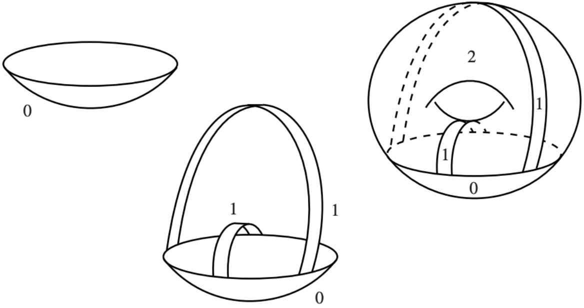
The diffeomorphism type of $M \cup_\varphi h$ is specified by:

1. an *embedding* $\varphi_0 : S^{k-1} = \partial D^k \times \{0\} \to \partial M$
2. a *(normal) framing* of $\varphi_0(S^{k-1})$, i.e., an identification of the normal bundle $TM|_{\varphi_0(S^{k-1})}/T\varphi_0(S^{k-1})$ with the trivial bundle $S^{k-1} \times \mathbb{R}^{n-k}$

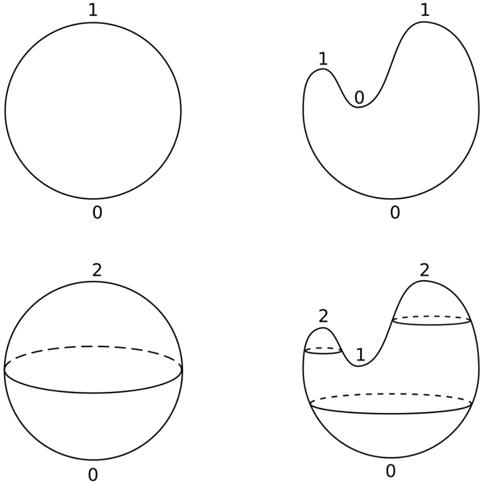


Handlebodies

If M is a compact n-manifold, then a handle decomposition of M is a way to obtain M by attaching handles. Every manifold admits a handle decomposition (Morse 1931). A manifold with a given handle decomposition is a handlebody.



Nonuniqueness of handle decompositions

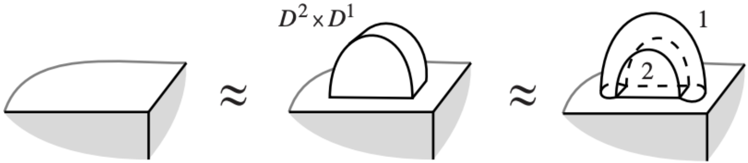


do not get mad at me for not having a hyphen in the nonword "nonuniqueness" i speak american english

Proposition

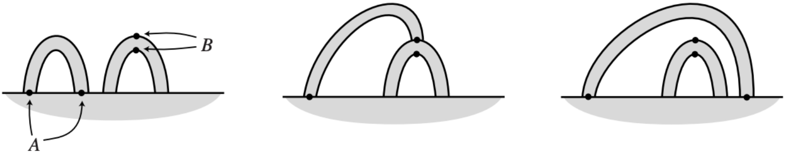
If h_{k-1} is a $(k-1)$-handle and h_k is a k-handle such that the attaching sphere A of h_k intersects the belt sphere B of h_{k-1} transversely at one point, then h_{k-1} and h_k can be canceled.

Requiring that A and B intersect transversely amounts to requiring that T_xA and T_xB span the tangent space of the ambient manifold, where x is the unique point in $A \cap B$.



Handle slides

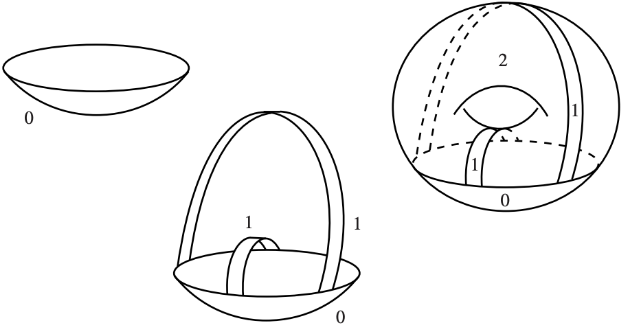
Consider k-handles h_1 and h_2 which are attached to ∂M. A handle slide is given by pushing the attaching sphere of h_1 through the belt sphere of h_2.

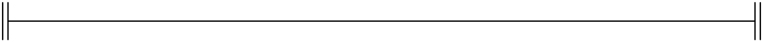


Theorem (Cerf 1970)

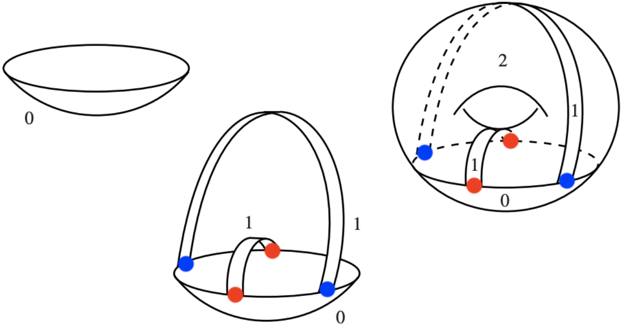
Any two handle decompositions of M can be made equivalent by sliding handles, creating or annihilating canceling handles, and isotoping within levels.

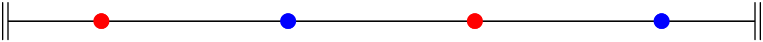
Drawing the torus in one dimension or something lol

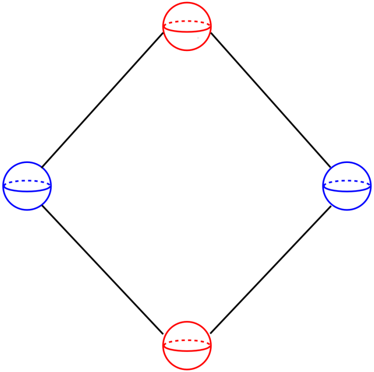




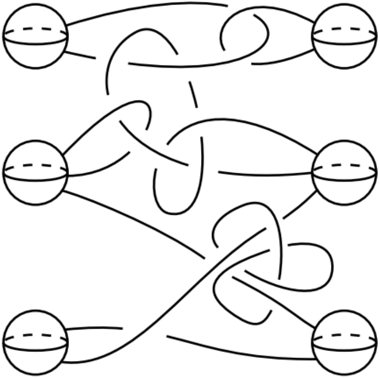
Drawing the torus in one dimension or something lol







A Kirby diagram of the cotangent bundle T^*T^2 of the torus



A Kirby diagram of something else entirely

Our UC Davis REU Experience

Section 2

Symplectic and Weinstein stuff

Symplectic forms

A differential k-form ω on M smoothly assigns a map

$$\omega_x : \underbrace{T_xM \times \cdots \times T_xM}_{k \text{ times}} \to \mathbb{R}$$

which is linear in each term for each $x \in M$ and which is alternating (e.g., for a 2-form, we always have $\omega_x(u, v) = -\omega_x(v, u)$).

There is a linear map d called the exterior derivative which takes k-forms to $(k+1)$-forms. It generalizes the differential of a function.

A symplectic form is a 2-form ω which is

1. *closed*: $d\omega = 0$
2. *nondegenerate*: If for any $v \in T_xM$, there exists $u \in T_xM$ such that $\omega_x(v, u) \neq 0$

If ω is a symplectic form on M, then we call (M, ω) a symplectic manifold. Symplectic manifolds are always even-dimensional!

Lagrangian submanifolds

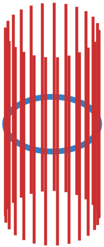
A submanifold X of a symplectic manifold (M, ω) is Lagrangian if, for every $x \in X$, we have $\omega_x|_{T_x X} \equiv 0$ and $\dim T_x X = \frac{1}{2} \dim T_x M$.

Example: There is a canonical symplectic form on T^*M. With this form, the zero section

$$M_0 = \{(x, \xi) \in T^*M : \xi = 0 \text{ in } T_x^*M\}$$

of T^*M is a Lagrangian submanifold of T^*M.

Example: The cotangent bundle over S^1 can also be visualized as the cylinder $S^1 \times \mathbb{R}$. In this case, the zero section is simply the blue copy of S^1.

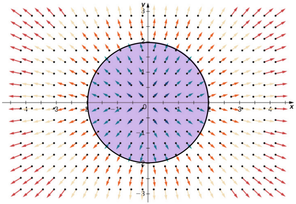


Liouville shenanigans

If $\omega = d\alpha$ for some 1-form α, then we call α a Liouville form. There is a vector field (i.e., a choice of tangent vector at every point $x \in M$) called the Liouville vector field associated to α.

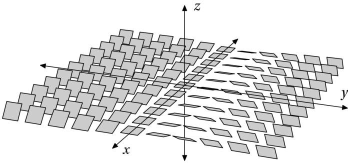
A Liouville domain is a symplectic manifold with boundary with a Liouville vector field that points transversely out of the boundary. Its skeleton is obtained by flowing the vector field backwards.

Example: There is a standard Liouville form on $\mathbb{R}^{2n}$. The Liouville vector field in this case is the radial vector field. The disk $D^{2n} \subset \mathbb{R}^{2n}$ is a Liouville domain whose skeleton is the origin.

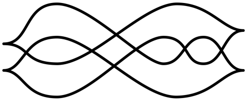


If M is a Liouville domain, then its Liouville form induces a contact structure on ∂M. In the case where M is a 4-manifold, a contact structure simply assigns a plane to every point on the boundary.

A knot in ∂M is an embedding $S^1 \to \partial M$. If this knot lies tangent to the contact structure at every point, then we call it Legendrian.

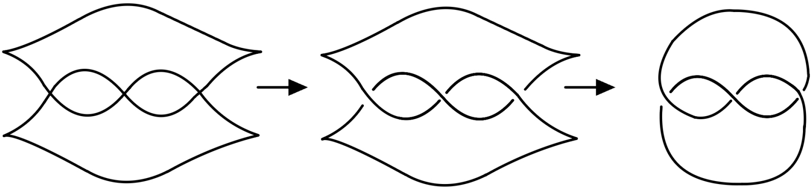


The standard contact structure in $\mathbb{R}^3$



An example of a Legendrian knot

A Legendrian knot is constrained by the contact structure in such a way that its y-coordinate is dz/dx. It's thus determined by its projection onto the xz-plane. This projection will be an immersion except at finitely many cusps and will have no vertical tangencies. The strand with more negative slope is in front.



Drawing a Legendrian trefoil smoothly

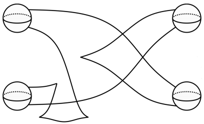
A Liouville domain is a Weinstein domain if there is an associated function which acts as a gradient.

In the symplectic case, we replace smooth k-handles with Weinstein k-handles.

Theorem (Weinstein 1991)

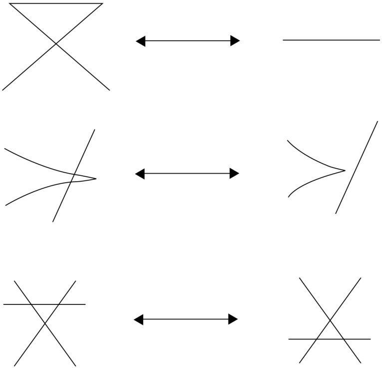
Any Weinstein $2n$-manifold can be decomposed into Weinstein k-handles for $0 \leq k \leq n$.

We can draw Kirby diagrams for Weinstein 4-manifolds (a diagram of the torus is shown below). In this setting, the attaching sphere of the 2-handle is a Legendrian knot whose framing is predetermined.

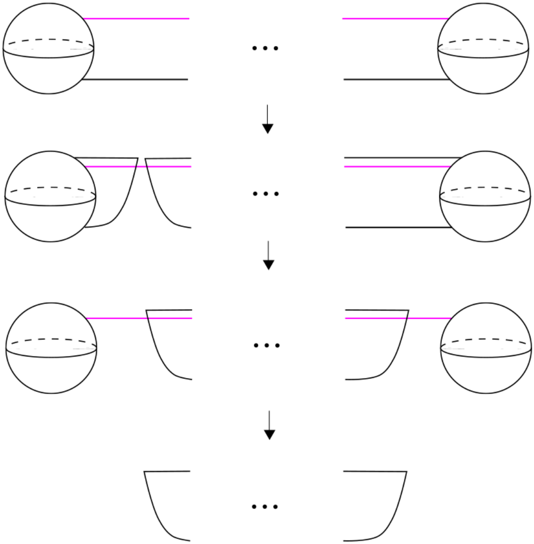


Theorem (Swiatkowski 1992)

Two front projections represent Legendrian isotopic knots if and only if the two diagrams can be related by a finite sequence of smooth isotopies and the Legendrian Reidemeister moves below.

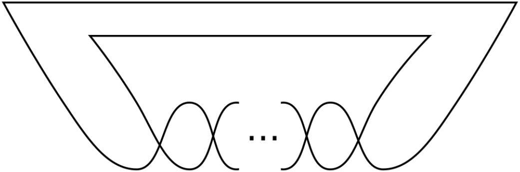


Handle slide rule cusp etc so on lalalala



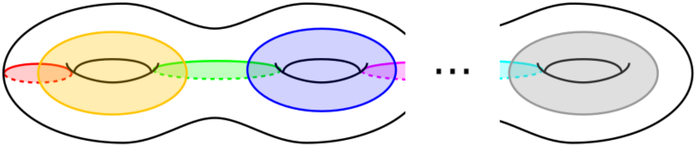
manifolds obtained by attaching 2 -handle along torus knots stuff

Consider the 4-manifold obtained by attaching a Weinstein 2-handle along the Legendrian $(2, n)$-torus knot.



While this manifold has a simple description, it is difficult to determine a sufficiently nice Lagrangian skeleton for it.

Now, consider the following arboreal Lagrangian skeleton obtained by attaching Lagrangian 2-disks to the (cotangent bundle) genus g surface.



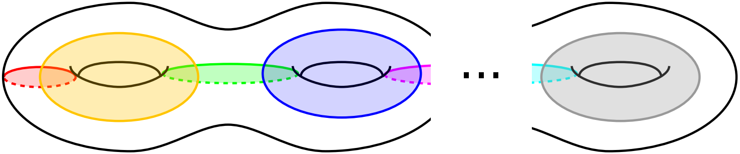
One of the main objectives of our project was to show that the resulting 4-manifold is the same as the 4-manifold defined by attaching a Weinstein 2-handle to a single 0-handle along the $(2, 2g+1)$-torus knot.

Our UC Davis REU Experience

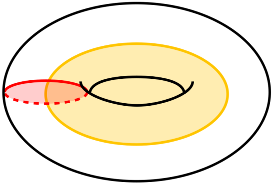
Section 3

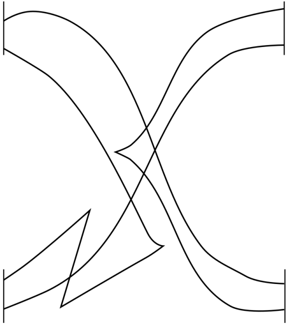
Results and stuff

Theorem. An arboreal Lagrangian skeleton for the 4-manifold obtained by attaching a Weinstein 2-handle along the $(2, 2g+1)$-torus knot to D^4 is given by the genus g surface with embedded disks:

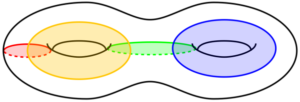


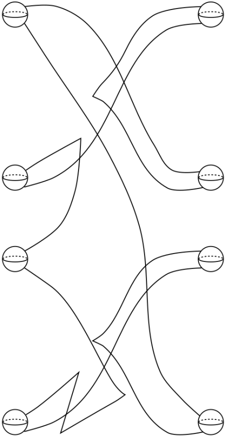
Genus 1:



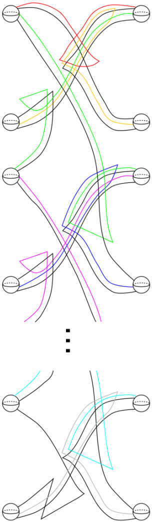


Genus 2:

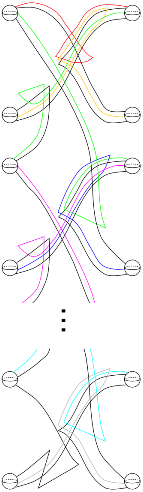


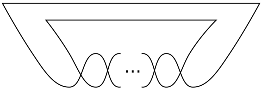


more holes

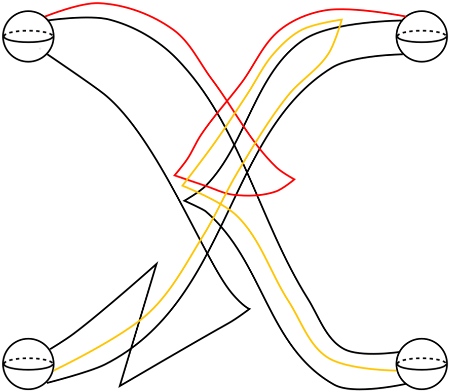


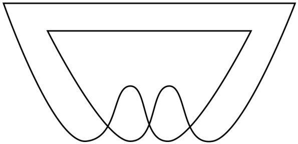
Our goal is to show that the Kirby diagrams below are equivalent.

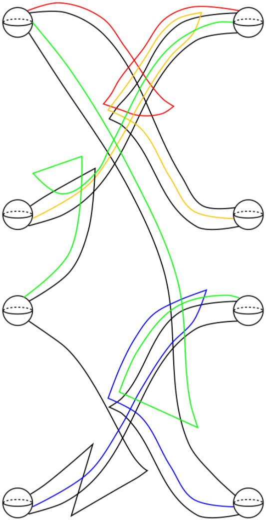


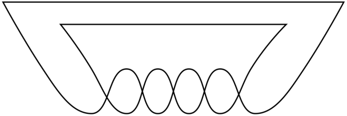


donut case

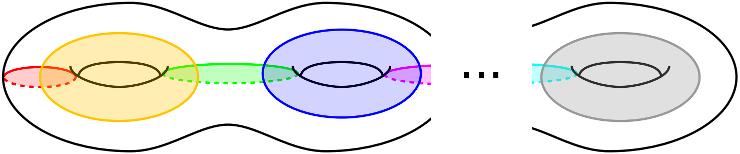








Theorem. An arboreal Lagrangian skeleton for the 4-manifold obtained by attaching a Weinstein 2-handle along the $(2, 2g+1)$-torus knot to D^4 is given by the genus g surface with embedded disks:



with a twist (or several)