Incompressible Fluids, Nontraditional Coriolis Terms, and Vortex Rings in Rotating Reference Frames

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Abstract

The full Coriolis force includes both the familiar sin-of-latitude terms as well as the frequently dropped cosine-of-latitude terms, known as the nontraditional Coriolis terms (NCT) [1]. In traditional atmospheric models, the NCT are often neglected as its effect in context of the momentum equations of fluids is insignificant for large scale flows. However, at the equator, the NCT reaches a maximum, thus raising the question of the validity of this traditional approximation. Using a poloidal model of circulation (DoNUT), Igel and Biello (2020) show that the nontraditional Coriolis force imparts a westward tilt to ascending equatorial convective flows, resulting in vertical upscale fluxes of zonal momentum on the mesoscale and ultimately revealing the importance of NCT in tropical convection. The purpose of our project was to experimentally verify this phenomenon using vortex rings in a rotating tank of water, as well as study the dynamics of vortex rings in general. Although we did not develop an explicit model of the background flow representing the westward convective tilt, we were still able to detect and record the circulation resulting from the toroidal vortices, setting foundation for future investigation. In the process of our project we studied the mathematics of incompressible fluids, rotational dynamics, and vortex rings.

1 Introduction

The Earth constitutes a rotating frame of reference over which the fluid of the atmosphere and ocean move. The Earth's rotation gives rise to the Coriolis effect, or Coriolis force -a pseudo force acting perpendicular to the direction of motion and to the axis of rotation in a rotating frame of reference. While it is well known that the Coriolis force dictates all atmospheric motion, its effect is most weakly experienced in the tropics. In a recent study, Igel and Biello (2020) have developed mathematical models that recognize the importance of the Coriolis force in organizing tropical thunderstorms.

Using a rotating fluid tank with a pair of small vortex cannons intended to mimic convective plumes in the tropical atmosphere, our experiment aimed to analyze the flow of the vortices as well as the dynamics of their interaction. With a variety of different setups and cannon parameters, we worked to collect data in order to find a pattern in the vortex ring induced circulation. Paralleling the theory established by Igel and Biello (2020), our hypothesis involved the detection of a mean flow, or background circulation, characterized by westward tilt in the ascending arm of the flow and eastward tilt in the descending arm. In the first part of this report, I will discuss some of the mathematics involved in our project, including rotating frames of reference and incompressible fluid dynamics. Then I will discuss the background and theoretical basis for our experiment in context of the atmosphere. Lastly I will go into the details of our experiment, involving the setup, procedure, data collection, and results.

2 Rotating Frames of Reference

To see where the Coriolis force arises from, one must consider the equations of motion in a rotating frame of reference where the direction of basis vectors changes over time. We'll let $\vec{\Omega} = \dot{\theta}$, $\vec{v} = \dot{r}$, and $\vec{a} = \dot{v}$

$$\frac{d\hat{i}}{dt} = \vec{\Omega} \times \hat{i}, \ \frac{d\hat{j}}{dt} = \vec{\Omega} \times \hat{j}, \ \frac{d\hat{k}}{dt} = \vec{\Omega} \times \hat{k}$$

Consider the rate of change of an arbitrary vector, $\vec{u} = u\hat{i} + v\hat{j} + w\hat{k}$, in $\mathbb{R}^3$,

$$\frac{d\vec{u}}{dt} = (\frac{\partial u}{\partial t}\hat{i} + \frac{\partial v}{\partial t}\hat{j} + \frac{\partial w}{\partial t}\hat{k}) + (u\frac{d\hat{i}}{dt} + v\frac{d\hat{j}}{dt} + w\frac{d\hat{k}}{dt}) = (\frac{\partial u}{\partial t}\hat{i} + \frac{\partial v}{\partial t}\hat{j} + \frac{\partial w}{\partial t}\hat{k}) + (u(\vec{\Omega} \times \hat{i}) + v(\vec{\Omega} \times \hat{j}) + w(\vec{\Omega} \times \hat{k}))$$

$$\implies (\frac{d\vec{u}}{dt})_I = (\frac{d\vec{u}}{dt})_R + \vec{\Omega} \times \vec{u} \tag{1}$$

The first term $(\frac{d\vec{u}}{dt})_I$ represents the time derivative of $\vec{u}$ in the stationary reference frame and $(\frac{d\vec{u}}{dt})_R$ represents the time derivative in the rotating frame.

Now consider an arbitrary position vector $\vec{r}$: $(\frac{d\vec{r}}{dt})_I = (\frac{d\vec{r}}{dt})_R + \vec{\Omega} \times \vec{r} \implies \vec{v}_I = \vec{v}_R + \vec{\Omega} \times \vec{r}$

Plug $\vec{v}_I$ into the general expression given by equation (1) :

$$\implies (\frac{d\vec{v}_I}{dt})_I = (\frac{d\vec{v}_I}{dt})_R + \vec{\Omega} \times \vec{v}_I = (\frac{d}{dt}(\vec{v}_R + \vec{\Omega} \times \vec{r}))_R + \vec{\Omega} \times (\vec{v}_R + \vec{\Omega} \times \vec{r})$$

$$\implies \vec{a}_I = \vec{a}_R + \frac{d}{dt}(\vec{\Omega} + \vec{r}_R) + \vec{\Omega} \times \vec{v}_R + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \quad = \quad \vec{a}_R + \vec{\Omega} \times \frac{d\vec{r}_R}{dt} + \vec{\Omega} \times \vec{v}_R + \vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

$$\therefore \ \vec{a}_I = \vec{a}_R + (2\vec{\Omega} \times \vec{v}_R) + (\vec{\Omega} \times (\vec{\Omega} \times \vec{r})) \tag{2}$$

Applying Newton's 2nd law, we'll multiply equation (2) by arbitrary mass, m. We have now reached an equation describing the forces felt by an object in a stationary reference frame in relation to a rotating frame:

$$\vec{F}_I = \vec{F}_R + 2m(\vec{\Omega} \times \vec{v}_R) + m\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \tag{3}$$

On the right hand side of equation (3), the 2nd and 3rd terms are interpreted as the Coriolis force and the centrifugal force, respectively. As shown, the Coriolis force acts in the direction of the cross product of the rotation vector, $\vec{\Omega}$, and the velocity in the rotating frame, $\vec{v}_R$, which is perpendicular to the direction of travel. [2]

3 Incompressible Fluid Dynamics

Foundational to the mathematics of fluid dynamics in the atmosphere is the concept of incompressible flow. The approximation of the atmosphere as an incompressible fluid turns out to be extremely accurate for the large scales in which motion takes place, simplifying models immensely.

Fluids are analyzed from two viewpoints: the Lagrangian/material view and the Eulerian/field view. The material view concerns analysis of fluid motion in terms of positions and momenta of particular fluid elements, while the field view concerns time evolution of the fluid field from a particular reference frame. These two viewpoints are related by the material derivative, defined as follows [2]:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \nabla$$

The material derivative describes the rate of change of a particular fluid element subject to a space and time dependent velocity field, $\vec{u}$. An incompressible fluid is a fluid in which $\frac{D\rho}{Dt} = 0$. This condition simplifies the mass continuity equation, yielding the familiar form:

$$\nabla \cdot \vec{u} = 0 \tag{4}$$

3.1 The Momentum Equation

The momentum equation is a partial differential equation describing how the velocity or momentum of a fluid responds to internal and imposed forces [2]. This PDE expresses conservation of momentum, essentially representing Newton's 2nd law for fluids. To derive the momentum equation, we'll first let $\vec{m}(x,y,z,t)$ be the momentum-density field of the fluid. Thus $\vec{m} = \rho\vec{u}$, and the total momentum of a volume of fluid can be expressed as, $\int_V \vec{m} dV$.

The rate of change of $\vec{m}$ of a fluid mass is given by the material derivative, and is equal to the forces acting upon it,

$$\frac{D}{Dt}\int_V \rho\vec{u} dV = \int_V \vec{F} dV \implies \int_V (\rho\frac{D\vec{u}}{Dt} - \vec{F}) dV = 0$$

Because the volume is arbitrary, the integrand must vanish $\implies \rho\frac{D\vec{u}}{Dt} = \vec{F}$. Expanding the material derivative then gives the general form of the momentum equation [2]:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} = \frac{\vec{F}}{\rho} \tag{5}$$

Inserting the pressure gradient and viscosity term, known as contact forces, and including (4) gives the incompressible Navier-Stokes equations:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} = -\nabla p + \nu\nabla^2\vec{u} + \frac{\vec{F}}{\rho}$$

$$\nabla \cdot \vec{u} = 0$$

Taking the curl of the Navier-Stokes momentum equation and applying several vector identities gives the analogous vorticity equation,

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{u} \cdot \nabla)\vec{\omega} - (\vec{\omega} \cdot \nabla)\vec{u} = \nu \nabla^2 \vec{\omega}$$

Vorticity, $\vec{\omega}$, defined as the curl of the velocity field, measures the local rotation of a fluid at some point. The second term on the left-hand side of the voritcity equation above represents the transport of vorticity by the velocity field. The third term, the vortex stretching term, represents the enhancement of vorticity by stretching and is important in the turbulence dynamics of the system [3]. The last term describes the effects of viscous diffusion on the vorticity distribution, resulting in the vorticity diffusing throughout space [3].

3.2 Stream Functions

Stream functions are an important concept in the analysis of fluids, defined for incompressible 2D fluids as well as 3D axisymmetric flows -such as poloidal flow. The stream function allows us to visualize streamlines that represent the flow of the fluid.

Consider $\vec{u} = u\hat{i} + v\hat{j}$. Define stream function, ψ, such that,

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}$$

Then, using the following,

$$\vec{u} = -\frac{\partial \psi}{\partial y}\hat{i} + \frac{\partial \psi}{\partial x}\hat{j}, \quad \nabla \psi = \frac{\partial \psi}{\partial x}\hat{i} + \frac{\partial \psi}{\partial y}\hat{j}$$

It is true that,

$$\vec{u} \cdot \nabla \psi = 0$$

Hence $\vec{u}$ is parallel to contour lines of constant ψ, or in other words, ψ is constant along the streamline of the flow. Also ψ satisfies the continuity equation since $\nabla \cdot \vec{u} = -\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial x \partial y} = 0$.

Using the stream function, vorticity can be expressed in the following way:

$$\vec{\omega} \equiv \nabla \times \vec{u} = \nabla \times (-\frac{\partial \psi}{\partial y}\hat{i} + \frac{\partial \psi}{\partial x}\hat{j}) = (\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2})\hat{k} = \nabla^2 \psi \tag{6}$$

Thus, ψ can be found from $\vec{\omega}$ using the Poisson's equation in (6). It should be noted that in the case of $\nabla^2 \psi = 0$, the fluid is irrotational. Combining equation (6) with the voriticity equation gives the stream function and voriticity formulation of the Navier-Stokes equation.

4 The Nontraditional Coriolis Force

Based upon the work of Igel and Biello (2020), the primary goal of our experiment was to test the theory associated with the dynamics of the circulation created by a poloidal flow in the presence of the Coriolis force -the organizer of planetary motion. However, at the equator, only the nontraditional Coriolis terms influence the motion associated with tropical convection.

To see where the nontraditional terms come from, we'll consider our equations of motion in a noninertial reference frame attached to the rotating Earth [1]. The momentum equation for an inviscid fluid in a rotating frame of reference is thus,

$$\frac{D\vec{u}}{Dt} + \nabla\vec{p} = -2\vec{\Omega} \times \vec{u} + \vec{F}_{ext} \tag{7}$$

with $\vec{u} = u\hat{i} + v\hat{j} + w\hat{k}$, where $\hat{i}$, $\hat{j}$, and $\hat{k}$ represent east, local north, and local up, respectively. We let $\vec{\Omega} = \Omega_0(cos\lambda\,\hat{j} + sin\lambda\,\hat{k})$, where Ω_0 is the rotation rate of the Earth and λ is the latitude. Expanding the cross product in the Coriolis force term and rewriting (7) in component form yields the equations:

$$\frac{Du}{Dt} + \frac{\partial p}{\partial x} = 2\Omega_0 v sin\lambda - 2\Omega_0 w cos\lambda$$

$$\frac{Dv}{Dt} + \frac{\partial p}{\partial y} = -2\Omega_0 u sin\lambda$$

$$\frac{Dw}{Dt} + \frac{\partial p}{\partial z} = 2\Omega_0 u cos\lambda + F_g$$

Here, the nontraditional Coriolis terms are the ones containing $cos\lambda$, as this quantity reaches a maximum at the equator, while the $sin\lambda$ terms are correspondingly reduced to zero.

To analyze the effects of the NCT, Igel and Biello (2020) define the net Coriolis force,

$$\vec{F}_{net} = \vec{F}_{tot} - \nabla p_C = -2\vec{\Omega} \times \vec{u} - \nabla p_C$$

where the Coriolis pressure, ∇p_C, serves to maintain a non-divergent net force, satisfying the condition that the fluid is incompressible [1]. A Leray projection is then used to diagnose this pressure field and subsequently find a general expression for the net, nondivergent force required to maintain a non-divergent flow, $\vec{u}$. Igel and Biello (2020) then apply this Leray projection to their DoNUT model, a poloidal model of convection, to arrive at an equation for the "Coriolis shear force",

$$\mathbf{F}_{CS}(0,z) = -2\Omega_0 cos(\phi)\frac{\partial G}{\partial y}\hat{i} = -\Omega_0 cos(\phi) w(z)\hat{i}$$

This force is shown to be proportional to the strength of the vertical velocity along the central axis and point westward in the ascending part of the convective flow. Igel and Biello (2020) show that this force corresponds to an upscale flux of zonal momentum to the east, causing westerly wind bursts on a larger scale.

5 The Experiment

By shooting vortex rings in a rotating tank, we were able to simulate a poloidal circulation in the presence of the Coriolis force. Our experiment allowed us to study the properties and dynamics of vortex rings, with the ultimate goal of detecting a mean flow associated with the larger scale circulation induced by the rings.

Our experiment took place in a cylindrical tank (1 meter diameter) filled with water about 12 centimeters high that rotates with a period of 36 seconds. The vortex rings were fired from two identical vortex cannons attached to a pressurized system involving a water reservoir, pistons, and a pressure tank. Controlled by Python code, we were able to adjust the parameters of the vortex cannons, allowing us to experiment with different pressures, valve openings, and durations of the firing mechanism.

In our set up, looking into the tank from above corresponds to standing at the equator and looking southward. With the tank rotating counterclockwise, vortex rings fired in the horizontal plane will naturally arc to the right, mimicking clouds rising in the atmosphere and drifting westward due to the Coriolis force.

5.1 Set Up

The following was done before the firing of each volley of vortex rings:

1. Connect the air pumper to the battery charger. Push the 220 V button on, which will be indicated by the white light.

2. Connect the water pump and the valve pressure machine with a dual adapter. Connect the dual adapter with the battery charger and turn the switch on.

3. Push the button that controls the pressure in the pressure tank. Turn the pressure to desired value (typically in the 80-95 psi range). Moniter this value regularly to ensure that it stays the same and adjust accordingly.

4. Adjust the pressure valve to the desired value.

5. Next, we fill up the water reservoir from the syringes. To do this, make sure that the blue knobs are turned towards the tubes attached to the cannons. After turning the knobs, push both syringes at the same time to fill the reservoirs. Make sure that there are no air bubbles in them. Moniter syringes after each experiment and replenish as necessary.

6. Fill the tank up with water to the depth of 12 centimeters.

7. Clean launching cannons *Alpha* and *Bravo* until they are clear of any residue.

8. Immerse the cannons in the water

9. Turn on the GoPro camera, making sure it is charged. Position it above the tank so that the entire system is in frame.

10. Turn the computer on and open up the program ‘vortex.py’ on the VNC Viewer app, which controls the launching of vortex rings from the cannon.

11. If everything above checks out, the experiment is ready to be performed.

5.2 Procedure

1. Check the pressure and valve, and ensure that the blue knobs of the water reservoirs are pointed up.

2. Check the duration and repeat parameters in the vortex.py file and adjust accordingly.

3. Next, to prepare the dye solution, fill two containers with 50 mL of water from the tank. Add 3 drops of methyl blue dye to each container, and stir well.

4. For each cannon, take it out of the water and push the aperture gently with a small metal rod until it can no longer be pushed back.

5. Pour 50 mL of the dye solution into the cannon through the aperture. If there is a significant amount of water that spills over, the cannon has not been pushed back all the way, so the dye solution should be dumped out, and the previous step should be repeated.

6. Start the switch that rotates the tank and set it to a period of 36 seconds. Wait for the tank to reach equilibrium.

7. Record the initial reservoir value, valve, pressure, duration, and repeat.

8. Start recording on the GoPro.

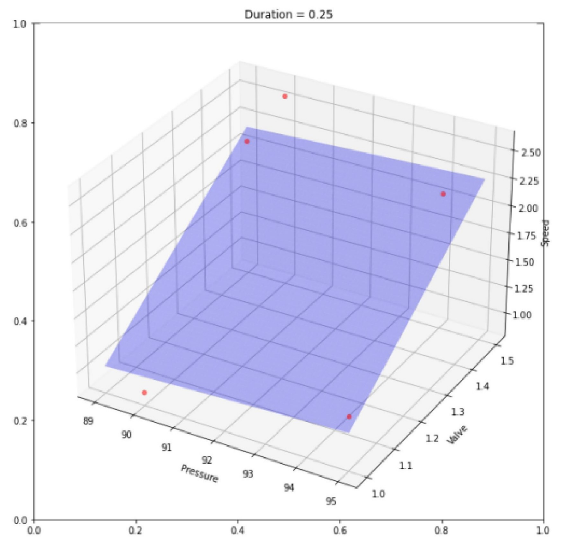
9. On the computer, run the code to launch the vortex rings.

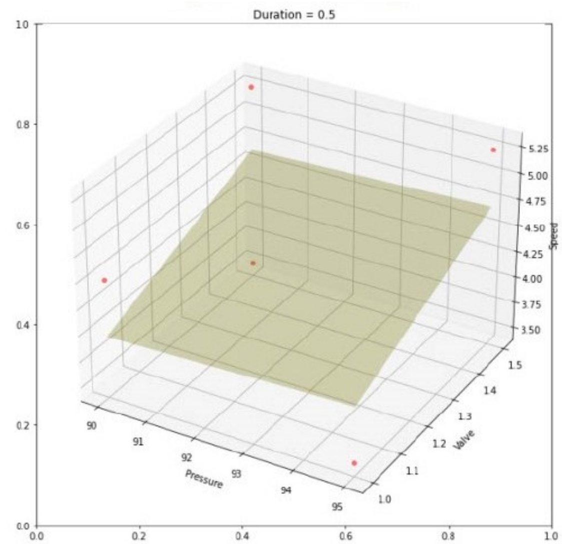
10. Record observations regarding approximate size, speed, and clarity of rings. Note any differences between the rings fired by the two different cannons. Record current reservoir value and pressure.

5.3 Analysis and Results

Before we could begin to analyze the ”macroscopic” motion in our tank, a program was required to track individual vortex rings so that we could explore properties such as size, shape, speed, and stability. The code for this program was developed by member of our team, Tyler Greiner, in MATLAB. Using recorded GoPro footage, we worked to process the data from our videos through this program to explore the relationships between the various parameters in our experiment.

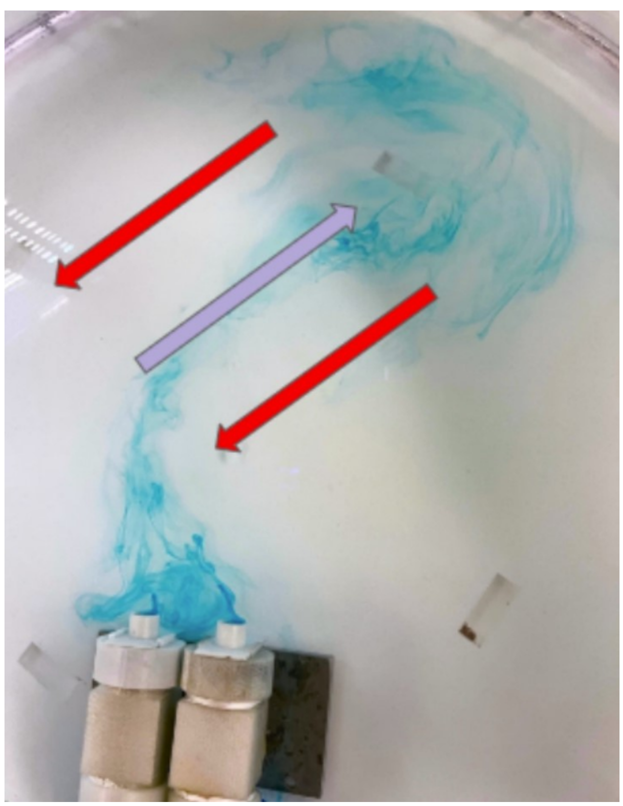
One important parameter of our experiment was the pressure valve, which was related to how much pressure passed through the system into the cannon. In order to understand this feature, we collected data of the vortex rings’ speed at different valve settings and different pressures. Two graphs showing the relationship between these parameters at different durations are shown below:





As shown, higher valve settings correspond to higher speeds. However, we found that beyond a certain valve threshold (around 2.5-3) the vortex rings began to deform quicker due to drag.

Tracking the individual vortex rings and processing data consumed the majority of our project so we did not have much time to analyze the macroscopic motion. However, by firing multiple vortex rings using specified parameters, we were able to observe the mean flow we were looking for:



The mean flow pictured above matched the theory established by Igel and Biello (2020), as rising flow tilted west and recirculation tilted east. With the methyl blue dye, we could visualize the mean flow but could not track its deformed shape. Thus, in order to properly analyze the macroscopic motion associated with the mean flow, a new method of appraoch was required.

As we determined, the dynamics of the mean flow can be best thought of as a vector field, so we needed a way to map out the velocity at each point in the tank. Our first approach involved using mica powder, composed of small, reflective two-dimensional sheets. By pouring the mica powder in the tank, we hoped to track the motion of the individual mica particles which would follow the flow. Although this method successfully allowed us to visualize the mean flow, tracking the mica proved to be a difficult task that we could not complete due to time constraints.

Despite lacking time to fully analyze the mean flow, our summer project succeeded in laying the experimental foundation for future investigation into the dynamics of rotating vortex rings. With sufficient code to track the mean flow, a stronger verification of the theory established by Igel and Biello (2020) can be attained.

References

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