# QUANTUM ERROR DETECTION AND LIE THEORY

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Abstract. Quantum metric spaces provide a framework that allows one to think of quantum error detection and correction problems geometrically rather than purely algebraically. We are especially interested in quantum metric spaces of Lie type – named so because they are constructed from representations of Lie algebras – which are highly symmetric quantum metric spaces roughly analogous to homogeneous metric spaces. We introduce the quantum error detection problem and give special attention to one method of solving it due to Knill, Laflamme, and Viola. Using this method, we demonstrate the existence of large error-detecting codes in certain quantum metric spaces of Lie type generated by representations of $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$. We also discuss and give examples of some even larger quantum codes that are constructed by slightly modifying Knill, Laflamme, and Viola's method.

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## 1. Introduction

Any classical (i.e. non-quantum) error correction or detection problem is equivalent to a sphere packing problem in a certain metric space. Thinking about packing spheres in a metric space is a very useful tool for understanding error correction problems. However, thinking about quantum error detection or correction problems in this way is more difficult, because standard (henceforth *classical*) metric spaces can't fully account for the complexities of the quantum setting. In [KW12], Kuperberg and Weaver defined *quantum metric spaces*, which are the objects that play the role analogous to classical metric spaces for quantum error correction and detection problems. That is to say, quantum metric spaces are equipped with a notion of distance that turns quantum error correction and detection problems into problems roughly analogous to sphere packing.

In both the classical and quantum cases, it's hard to say very much about error detection problems in general spaces. The definition of a metric space is extremely broad, and this allows for the existence of many pathological spaces that are difficult to work in. The same is true of quantum metric spaces. To get around this, it's common to restrict one's attention to highly symmetric metric spaces. The analogous construct in the quantum case is the notion of quantum metric spaces of Lie type. These are quantum metric spaces constructed from representations of Lie algebras, and their symmetry properties make them much more favorable for studying error detection problems.

The sections of this paper are organized as follows. In Section 2, we introduce classical error correction on classical metric spaces to motivate the quantum case. We then introduce quantum metric spaces in Section 3, and explore some basic properties of these spaces. In Section 4, we define and examine quantum metric spaces of Lie type. In Section 5, we introduce some basic quantum probability in order to cover the theory of quantum codes and quantum error detection in these quantum metric spaces. In Section 6, we cover

the method of Knill, Laflamme, and Viola in [KLV00] for constructing quantum codes in general quantum metric spaces. We demonstrate that in certain quantum metric spaces of Lie type, we can exploit symmetry properties to find much larger quantum codes than the bound given in [KLV00] would suggest. In particular, for quantum metrics spaces constructed from irreducible string representations of $\mathfrak{su}(3)$, we construct codes of dimension roughly 4/27 the dimension of the whole space. In Section 7, we give a brief look at some quantum codes that aren't constructed with KLV's method.

## 2. Classical error correction

Many classical error correction problems may be posed as follows. We start with a metric space $(X, d)$ thought of as a space of messages, where the distance reflects how similar two messages are to each other. We assume errors take messages to nearby ones. A *code* is simply a subset $C \subseteq X$, and its *minimum distance* $d(C)$ is defined as

$$d(C) = \inf_{\substack{x,y \in C \\ x \neq y}} d(x, y)$$

If Alice sends a message to Bob from the code $C$ and an error of size less than $d(C)$ occurs, then the message Bob receives will not be in $C$, since no element of $C$ is within this distance of the original message. We say such an error is *detectable*. Moreover, if an error of size less than $d(C)/2$ occurs, then Bob can determine the original message by simply rounding to the nearest element of $C$. Such an error is said to be *correctable*. Error correction and detection problems center on questions like the following: If we are looking for a code of a given size, how large can its minimum distance be? Conversely, for a fixed minimum distance, how big of a code can we find?

Supposing code $C$ corrects $t$ errors, if we place a ball of radius $t$ at each $x \in X$, the triangle inequality implies these balls will be disjoint. Indeed, finding a code in $X$ that corrects $t$ errors is equivalent to packing spheres of radius $t$ in $X$.

In general metric spaces, it's quite difficult to make much progress on problems of this sort. More progress can be made in metric spaces which are *homogeneous*, meaning for any $x, y \in X$, there exists an isometry of $X$ taxing $x$ to $y$. Heuristically, these are the metric spaces that "look the same" at each point – each error ball is the same shape. Even more favorable are *2-point symmetric* metric spaces, meaning for any two pairs $(x_1, x_2), (y_1, y_2) \in X^2$ such that $d(x_1, x_2) = d(y_1, y_2)$, there is an isometry of $X$ taking $x_1$ to $y_1$ and $x_2$ to $y_2$ simultaneously.

In computer science contexts, this problem is most often studied in Hamming space, where the space $X = \{0, 1\}^n$ consists of bit strings of a fixed length $n$, and the distance between two bit strings is the number of positions in which they differ. If instead we let $X$ be $\mathbb{R}^n$ with the Euclidean metric, the problem becomes classical sphere packing.

The following example will prove useful later.

**Example 2.1.** Consider the metric space given by

$$\Delta(n, k) = \left\{ (x_1, \ldots, x_k) \in \mathbb{Z}^k \;\middle|\; x_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^{k} x_i = n \right\},$$

and

$$d((x_1, \ldots, x_k), (y_1, \ldots, y_k)) = \frac{1}{2} \sum_{i=1}^{k} |x_i - y_i|.$$

The condition $\sum_{i=1}^{k} x_i = n$ implies that all points in $\Delta(n, k)$ lie in a $(k-1)$-dimensional hyperplane. The points form a $(k-1)$-dimensional simplicial lattice,[1]. In fact, $d$ is a taxicab-like metric on this lattice, meaning the distance between any two points in $\Delta(n, k)$ is the number of distance-1 steps one must take to get between them.

A minimum distance 2 subset of $\Delta(n, k)$ is

$$T(n, k) = \left\{ (x_1, \ldots, x_k) \in \Delta(n, k) \;\middle|\; \sum_{i=1}^{k} i x_i = 0 \pmod{k} \right\}$$

[1]More precisely, this is a subset of the well-studied $A_k$ lattice, also called a *simplectic honeycomb* shown in figure 1. In two dimensions it is an equilateral triangular lattice, and in three dimensions it is a regular tetrahedral lattice.

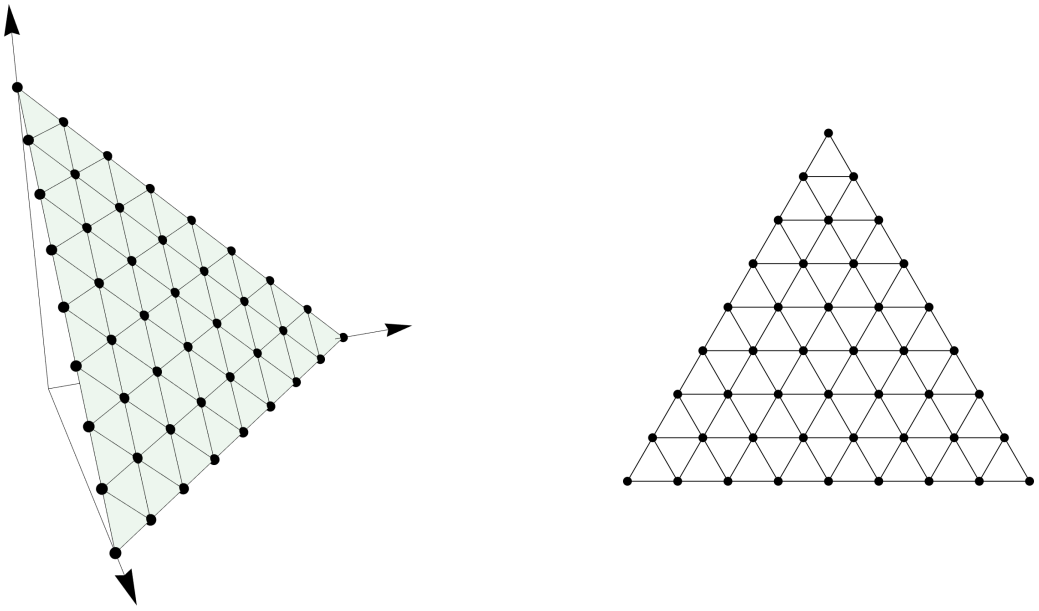


FIGURE 1. $\Delta(8,3)$ shown in $\mathbb{R}^3$, and its projection into $\mathbb{R}^2$. The distance between any two points is the minimum number of edges one must traverse to get between them in the diagram.

It isn't too difficult to see that no two elements of this subset are adjacent, so the minimum distance is indeed 2.

## 3. Finite-dimensional quantum metric spaces

Quantum error correction problems do not initially appear analogous to classical ones. The notion of a quantum metric space, (or $W^*$ metric space), first defined in [KW12], bridges this divide, so that finding error correcting codes reduces to packing error balls in quantum metric spaces.

Let $\mathcal{H} \cong \mathbb{C}^d$ be a finite-dimensional complex Hilbert space, and let $\mathcal{L}(\mathcal{H}) \cong M_d(\mathbb{C})$ denote the set of linear operators from $\mathcal{H}$ to itself. $\mathcal{H}$ is interpreted as the state space of a quantum system, and $\mathcal{L}(\mathcal{H})$ is interpreted as the space of errors acting on states.

**Definition 3.1.** A *quantum pseudometric* on $\mathcal{H}$ is a $*$-algebra filtration $\{\mathcal{V}_t\}$ – that is, a vector subspace $\mathcal{V}_t$ of $\mathcal{L}(\mathcal{H})$ for each $t \in [0,\infty)$ – satisfying

(1) $I \in \mathcal{V}_0$
(2) $\mathcal{V}_t = \mathcal{V}_t^*$
(3) $\mathcal{V}_s\mathcal{V}_t \subseteq \mathcal{V}_{s+t}$
(4) $\mathcal{V}_t = \bigcap_{s>t} \mathcal{V}_s$

A quantum pseudometric is said to be a *quantum metric* if $\mathcal{V}_0 = \operatorname{span}\{I\}$. The parameter $t$ is thought of as cataloging the severity of errors: If $s > t$, elements of $\mathcal{V}_s$ are viewed as more severe errors than elements of $\mathcal{V}_t$.

In the definition above, $t$ can vary continuously from 0 to $\infty$, but in the finite-dimensional case this is somewhat misleading. Consider a $*$-algebra filtration $\{\mathcal{V}_t\}$ of $\mathcal{H} \cong \mathbb{C}^d$. Note that $\dim \mathcal{L}(\mathcal{H}) = d^2$. For each $t \geq 0$, $\dim \mathcal{V}_t$ must be an integer between 1 and $d^2$. Hence, there are a finite number of times $\mathcal{V}_t$ jumps up in dimension as $t$ increases. In practice, we only need to worry about these values of $t$, as for all other values of $t$, $\mathcal{V}_t$ is equal to $\mathcal{V}_s$ for some $s < t$. For most quantum metric spaces we work with, these jumps occur at integer values of $t$, so we only worry about some finite chain of subspaces $\mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_n$.

Given a quantum (pseudo-)metric $(\mathcal{H}, \{\mathcal{V}_t\})$, we can define a *displacement gauge* $D\colon \mathcal{L}(\mathcal{H}) \to [0,\infty]$ by

$$D(X) = \min\{t \in [0,\infty] \mid X \in \mathcal{V}_t\},$$

where we follow the convention $\min \varnothing = \infty$. Note that condition (4) of definition 3.1 implies that such a minimum always exists. The properties of a quantum pseudometric imply $D$ satisfies

(1) $D(XY) \leq D(X) + D(Y)$
(2) $D(X+Y) \leq \max\{D(X), D(Y)\}$
(3) $D(X^*) = D(X)$
(4) $D(\alpha X) = D(X)$ if $\alpha \neq 0$
(5) $D(I) = 0$

If $(\mathcal{H}, \{\mathcal{V}_t\})$ is a quantum metric space, (5) may be replaced by the stronger condition

(5') $D(X) = 0$ if and only if $X = \alpha I$ for some $\alpha \in \mathbb{C}$

One can recover the $*$-algebra filtration from its corresponding displacement gauge via the equation

$$\mathcal{V}_t = \{X \in \mathcal{L}(\mathcal{H}) \mid D(X) \leq t\}.$$

Thus, one may define a quantum metric space either in terms of a $*$-algebra filtration, or a displacement gauge. We will pass between these definitions interchangeably.

**Example 3.2.** Quantum Hamming space on $n$ qubits has $\mathcal{H} = (\mathbb{C}^2)^{\otimes n} \cong \mathbb{C}^{2^n}$, which implies $\mathcal{L}(\mathcal{H}) = M_2(\mathbb{C})^{\otimes n} \cong M_{2^n}(\mathbb{C})$. For each integer $0 \leq t \leq n$, we have

$$\mathcal{V}_t = \text{span}\{X_1 \otimes X_2 \otimes \cdots \otimes X_n \mid X_1, \ldots, X_n \in M_2(\mathbb{C}),\ X_i \neq I \text{ for at most } t \text{ terms}\}$$

The corresponding displacement gauge is defined on pure tensors by

$$D(X_1 \otimes X_2 \otimes \cdots \otimes X_n) = \text{ number of } X_i\text{'s not equal to the identity,}$$

and extended to the rest of the space via properties (2) and (4).

Just as in the classical case, an isometry of a quantum metric space is a map from the space to itself that preserves the metric. Precisely, the group of isometries is given by

$$\text{Isom}(\mathcal{H}, \{\mathcal{V}_t\}) = \{U \colon \mathcal{H} \to \mathcal{H} \mid U \text{ is unitary, and } D(UXU^*) = D(X) \text{ for all } X \in \mathcal{H}\}$$

We say a quantum metric space $(\mathcal{H}, \{\mathcal{V}_t\})$ is *connected* if $D(X)$ is finite for all $X \in \mathcal{L}(\mathcal{H})$. Equivalently, $(\mathcal{H}, \{\mathcal{V}_t\})$ is connected if and only if there is some $t \in [0, \infty)$ such that $\mathcal{V}_t$ is all of $\mathcal{L}(\mathcal{H})$. In this case, we call this $t$ the *diameter* of $(\mathcal{H}, \{\mathcal{V}_t\})$.

## 4. Quantum metrics of Lie type

We first define an very broad class of quantum metric spaces which are quantum analogs of graphs.

**Definition 4.1.** Let $\mathcal{H} = \mathbb{C}^d$. Any subspace $\mathcal{E}$ of $\mathcal{L}(\mathcal{H}) \cong M_d(\mathbb{C})$ such that $I \in \mathcal{E}$ and $\mathcal{E} = \mathcal{E}^*$ is called an *error space*.

**Definition 4.2.** Let $\mathcal{H} = \mathbb{C}^d$ and let $\mathcal{E}$ be an error space. A quantum metric space of the form

$$\mathcal{V}_0 = \text{span}_{\mathbb{C}}\{I\}, \qquad \mathcal{V}_1 = \mathcal{E}, \qquad \mathcal{V}_n = \text{span}_{\mathbb{C}} \underbrace{\mathcal{E}\mathcal{E}\cdots\mathcal{E}}_{n} \text{ for each integer } n \geq 2,$$

is called a *quantum graph metric*. In this context, we refer to $\mathcal{E}$ as the *edge space* of the quantum metric.

Calling these "quantum graph metrics" is justified, since quantum metrics of this form are analogous to the classical metrics on graphs given by path distance. In a graph, one specifies the set of points of distance 1, and the distance between any two points is the minimum number of distance 1 steps separating them. The same procedure is used in the quantum analog.

**Example 4.3.** Quantum Hamming space, defined in example 3.2 is a quantum graph metric with edge space

$$\mathcal{E} = \text{span}\{X_1 \otimes X_2 \otimes \cdots \otimes X_n \mid X_1, \ldots, X_n \in M_2(\mathbb{C}),\ X_i = I \text{ for all except 1 term}\}.$$

We now introduce quantum metrics of Lie type, as discussed in [Bum11].

**Definition 4.4.** Let $\mathfrak{g}_{\mathbb{R}}$ be a Lie algebra with base field $\mathbb{R}$, and let $\mathfrak{g}_{\mathbb{C}}$ be its complexification. Let $(\mathcal{H}, \rho)$ be a representation of $\mathfrak{g}_{\mathbb{R}}$ such that $\rho(\mathfrak{g}_{\mathbb{R}})$ is traceless and antihermitian. Note that $(\mathcal{H}, \rho)$ may be extended to a representation of $\mathfrak{g}_{\mathbb{C}}$. If we construct a quantum graph metric with edge space $\mathcal{E} = \text{span}_{\mathbb{C}}\{I\} \oplus \rho(\mathfrak{g}_{\mathbb{C}})$, the resulting metric is said to be of *Lie type*.

Analogous to many constructions in algebra, there exist both internal and external descriptions of Lie type quantum metric spaces. Above we have given the external description, where we construct a quantum metric starting from a representation of a Lie algebra. The following proposition presents an internal description – i.e. a condition for a quantum metric space being of Lie type.

**Proposition 4.5.** Suppose $(\mathcal{H}, \{\mathcal{V}_t\})$ is a quantum graph metric space with edge space $\mathcal{E}$. Then $(\mathcal{H}, \{\mathcal{V}_t\})$ is of Lie type if and only if $\mathcal{E}$ is closed under the matrix Lie bracket.

*Proof.* The forward direction is clear. Suppose that $\mathcal{E}$ is closed under the matrix Lie bracket. Let

$$\mathfrak{g}_{\mathbb{C}} = \{X \in \mathcal{E} \mid \operatorname{tr}(X) = 0\} = \mathcal{E} \cap \mathfrak{sl}(d, \mathbb{C}),$$

and

$$\mathfrak{g}_{\mathbb{R}} = \{X \in \mathcal{E} \mid \operatorname{tr}(X) = 0,\ X^* = -X\} = \mathcal{E} \cap \mathfrak{su}(d).$$

Note that $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{R}}$ are each Lie algebras since they are given by the intersection of two Lie algebras. Then $\mathcal{H}$ is a representation of $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{g}_{\mathbb{C}}$ with representation map $\rho(X) = X$. By construction, $\rho(\mathfrak{g}_{\mathbb{R}})$ is traceless and antihermitian and $\mathcal{E} = \operatorname{span}_{\mathbb{C}}\{I\} \oplus \rho(\mathfrak{g}_{\mathbb{C}})$ as desired. □

Hence, we may equivalently define Lie type quantum metrics as those quantum graph metric spaces for which the edge space is a Lie algebra.

**Example 4.6.** Quantum Hamming space is of Lie type, which can be verified by checking that the edge space $\mathcal{E}$ of example 4.3 is closed under the Lie bracket. To see this more explicitly, we consider the Lie algebra

$$\mathfrak{g}_{\mathbb{R}} = \underbrace{\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \cdots \oplus \mathfrak{su}(2)}_{n}$$

and its complexification

$$\mathfrak{g}_{\mathbb{C}} = \underbrace{\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \cdots \oplus \mathfrak{sl}(2, \mathbb{C})}_{n}.$$

The representation of interest has $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$ and is given by

$$\rho(X_1 \oplus X_2 \oplus \cdots \oplus X_n) = X_1 \otimes I \otimes \cdots \otimes I + I \otimes X_2 \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes X_n.$$

It's clear that $\operatorname{span}_{\mathbb{C}}\{I\} \oplus \rho(\mathfrak{g}_{\mathbb{C}})$ yields the edge space $\mathcal{E}$ of example 4.3.

**Example 4.7.** Consider the following representation of $\mathfrak{su}(d)$ (and its complexification $\mathfrak{sl}(d, \mathbb{C})$). Let $\mathcal{H}_n = \mathbb{C}[x_1, \ldots, x_d]_n$ denote the space of homogeneous polynomials of degree $n$ in $x_1, \ldots, x_d$. For a matrix $A = (a_{ij}) \in \mathfrak{sl}(d) \subseteq M_d(\mathbb{C})$, define

$$\rho(A) = \sum_{1 \le i,j \le d} a_{ij} x_j \frac{\partial}{\partial x_i}.$$

This is an irreducible representation of $\mathfrak{su}(d)$ and $\mathfrak{sl}(d, \mathbb{C})$.

First consider the $d = 2$ case. We have $\mathcal{H}_n = \operatorname{span}_{\mathbb{C}}\{x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n\}$. $\mathfrak{sl}(2, \mathbb{C})$ is spanned by the matrices

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

which get mapped to

$$\rho(E) = y\frac{\partial}{\partial x}, \qquad \rho(F) = x\frac{\partial}{\partial y}, \qquad \rho(H) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

The effect of each of these operators is shown in the following diagram.

$$\underset{\circlearrowleft 3}{x^3} \underset{1}{\overset{3}{\rightleftarrows}} \underset{\circlearrowleft 1}{x^2y} \underset{2}{\overset{2}{\rightleftarrows}} \underset{\circlearrowleft -1}{xy^2} \underset{3}{\overset{1}{\rightleftarrows}} \underset{\circlearrowleft -3}{x^2}$$

In order to build a quantum metric space from this representation, we first need to put a Hilbert space structure $\langle \cdot, \cdot \rangle$ on $\mathcal{H}_n$. The most natural way to do this is to make $\{x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n\}$ an orthogonal set, and to require that $\rho(E)^* = \rho(F)$. This means

$$\left\langle \rho(E)x^k y^{n-k}, x^{k-1}y^{n-k+1} \right\rangle = \left\langle x^k y^{n-k}, \rho(F)x^{k-1}y^{n-k+1} \right\rangle,$$

which implies

$$\left\langle x^{k-1}y^{n-k+1}, x^{k-1}y^{n-k+1}\right\rangle = \frac{n-k+1}{k}\left\langle x^k y^{n-k}, x^k y^{n-k}\right\rangle.$$

Up to a global constant factor, this implies

$$\left\langle x^k y^{n-k}, x^k y^{n-k}\right\rangle = \binom{n}{k}.$$

Hence, we define $|k, n-k\rangle = \frac{1}{\sqrt{\binom{n}{k}}} x^k y^{n-k}$, and impose the inner product on $\mathcal{H}_n$ in which $\{|k, n-k\rangle \,|0 \leq k \leq n\}$ is an orthonormal basis. In this new basis, we can compute

$$\begin{aligned}
\rho(E)\,|k, n-k\rangle &= \sqrt{k(n-k+1)}\,|k-1, n-k+1\rangle \\
\rho(F)\,|k, n-k\rangle &= \sqrt{(k+1)(n-k)}\,|k+1, n-k-1\rangle \\
\rho(H)\,|k, n-k\rangle &= (n-2k)\,|k, n-k\rangle
\end{aligned}$$

Our diagram now looks like this:

$$\begin{array}{ccccccc}
|30\rangle & \overset{\sqrt{3}}{\longleftrightarrow} & |21\rangle & \overset{2}{\longleftrightarrow} & |12\rangle & \overset{\sqrt{3}}{\longleftrightarrow} & |03\rangle \\
\circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft \\
3 & & 1 & & -1 & & -3
\end{array}$$

Note that we've combined the arrows for $E$ and $F$, since their respective coefficients are the same in this normalization. This yields a quantum metric space of Lie type with

$$\mathcal{V}_1 = \mathcal{E} = \mathrm{span}_{\mathbb{C}}\{I\} \oplus \rho(\mathfrak{sl}(2, \mathbb{C})).$$

In particular, the distance one errors are the ones shown by arrows in the diagram.

We can go through an analogous process for $\mathfrak{su}(d)$ with larger values of $d$. In particular, we define the basis vectors

$$|a_1 \cdots a_d\rangle = \frac{1}{\sqrt{\binom{n}{a_1, \ldots, a_d}}} x_1^{a_1} \cdots x_d^{a_d},$$

where $a_1 + \cdots + a_d = n$, and give $\mathcal{H}_k$ an inner product where this basis is orthonormal. We may build a Lie type quantum metric $(\mathcal{H}_n, \{\mathcal{V}_t\})$ where

$$\mathcal{V}_1 = \mathcal{E} = \mathrm{span}_{\mathbb{C}}\{I\} \oplus \rho(\mathfrak{sl}(d, \mathbb{C})).$$

In particular, distance 1 errors are either diagonal, or they take vectors to adjacent ones in the diagram in figure 2. Note that basis vectors for $\mathcal{H}_n$ are indexed by elements of $\Delta(n, d)$, and that the diagram in figure 2 looks very similar to the metric space $\Delta(n, d)$ in figure 1.

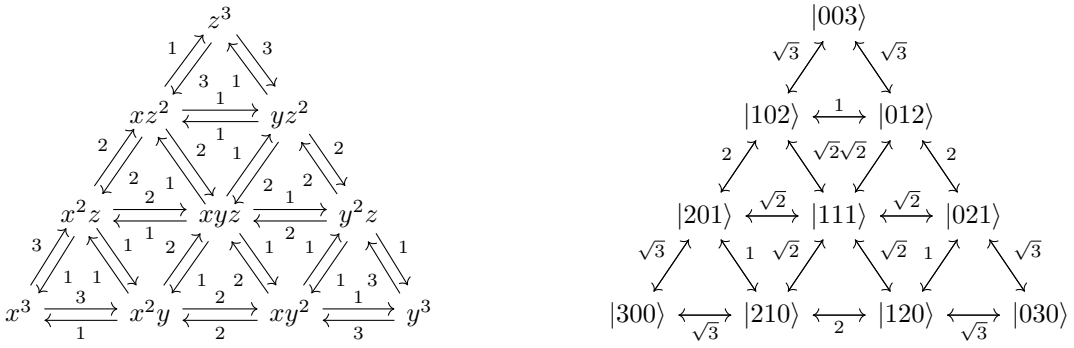



FIGURE 2. Diagrams for representation $\mathcal{H}_3$ of $\mathfrak{su}(3)$, unnormalized on left and normalized on right. Diagonal operators are not shown.

One motivation for studying quantum metric spaces of Lie type is that they have much more symmetry than general quantum metric spaces. In fact, their isometry groups are very closely related to the Lie algebras they are built from.

Suppose $A \in M_n(\mathbb{C})$ is an isometry of $(\mathcal{H}, \{\mathcal{V}_t\})$. Hence $A$ is unitary and $A\mathcal{V}_1A^{-1} = \mathcal{V}_1$. As $\operatorname{tr}\big(AXA^{-1}\big) = \operatorname{tr}(X)$, $A\mathfrak{g}_{\mathbb{C}}A^{-1} = \mathfrak{g}_{\mathbb{C}}$. And, since $A$ is unitary, $(AXA^{-1})^* = AX^*A^{-1}$. Hence, $X$ is anti-hermitian if and only if $AXA^{-1}$ is. Thus $A\mathfrak{g}_{\mathbb{R}}A^{-1} = \mathfrak{g}_{\mathbb{R}}$. Thus, elements of $\operatorname{Isom}(\mathcal{H}, \{\mathcal{V}_t\})$ each yield automorphisms of $\mathfrak{g}_{\mathbb{R}}$ by conjugation.

For a matrix $X$, define the map $\Psi_X \colon \mathfrak{g}_{\mathbb{R}} \to \mathfrak{g}_{\mathbb{R}}$ by $\Psi_X(Y) = XYX^{-1}$. By our observation above $\Psi$ is a homomorphism from $\operatorname{Isom}(\mathcal{H}, \{\mathcal{V}_t\})$ to $\operatorname{Aut}(\mathfrak{g}_{\mathbb{R}})$. If $G$ denotes the lie group corresponding to $\mathfrak{g}_{\mathbb{R}}$, then $\Psi$ gives the well-known adjoint representation Ad of $G$, which is a homomorphism from $G$ into $\operatorname{Aut}(\mathfrak{g}_{\mathbb{R}})$. The fact that $\Psi$ yields a homomorphism from both $G$ and $\operatorname{Isom}(\mathcal{H}, \{\mathcal{V}_t\})$ into $\operatorname{Aut}(\mathfrak{g}_{\mathbb{R}})$ tells us that these groups are very closely related. Often, these groups coincide exactly, or one is a large subgroup of the other.

Below is another nice property of quantum metric spaces of Lie type that relates their geometry to their algebraic structure.

**Theorem 4.8.** *Suppose $(\mathcal{H}, \{\mathcal{V}_t\})$ is a quantum metric space of Lie type with underlying Lie algebra $\mathfrak{g}_{\mathbb{R}}$ and representation map $\rho$. $(\mathcal{H}, \{\mathcal{V}_t\})$ is connected if and only if $\mathcal{H}$ is an irreducible representation of $\mathfrak{g}_{\mathbb{R}}$.*

In both the classical and quantum cases, error correction problems in disconnected metric spaces are less interesting. Finding codes in disconnected spaces involves finding codes in each of the connected components separately. In light of this the theorem above, we will restrict our attention to those in which the underlying representation is irreducible.

## 5. Quantum codes and error detection

In quantum probability, a boolean measurement – i.e. a question of the system with a yes or no answer – is specified by an orthogonal projection $P \colon \mathcal{H} \to \mathcal{H}$. If $\mathcal{C}$ is the image of $P$, the boolean measurement asks whether a state $|\psi\rangle$ is in $\mathcal{C}$ or not. Given a normalized state $|\psi\rangle \in \mathcal{H}$, the probability of "yes" is the squared magnitude of the component of $\psi$ in $\mathcal{C}$. That is,

$$\Pr[\text{yes}] = \langle\psi| P |\psi\rangle .$$

The resulting (unnormalized) state is $P|\psi\rangle$. It's customary to rescale this state to obtain a normalized one, but for our purposes this isn't necessary. Similarly, the probability of "no" is the squared magnitude of the component of $|\psi\rangle$ in the orthogonal complement of $\mathcal{C}$, and resulting unnormalized state is the projection of $|\psi\rangle$ onto that orthogonal complement, (i.e. $(I - P)|\psi\rangle$).

A quantum code $\mathcal{C}$ is simply a subspace of $\mathcal{H}$, which we interpret as the space of messages we can send. Let $P_{\mathcal{C}} \in \mathcal{L}(\mathcal{H})$ denote the orthogonal projection onto this subspace. The basic procedure for error detection is as follows. Suppose Alice wish to send the state $|\psi\rangle \in \mathcal{C}$ to Bob, and an error $E \in \mathcal{L}(\mathcal{H})$ occurs so that $E|\psi\rangle$ is received. Bob applies the boolean measurement associated with $P_{\mathcal{C}}$ to determine whether his received message is in the $\mathcal{C}$. If he measures "no," then the error is detected. Otherwise, if he measures "yes," the resulting state is $P_{\mathcal{C}}E|\psi\rangle$, which must be proportional to $|\psi\rangle$ for the correct message to be received. Thus, $\mathcal{C}$ detects $E$ if

$$P_{\mathcal{C}}E|\psi\rangle \propto |\psi\rangle$$

for all $|\psi\rangle \in \mathcal{C}$. An equivalent formulation is that $\mathcal{C}$ detects an error $E$ if there exists an $\varepsilon \in \mathbb{C}$ such that

$$P_{\mathcal{C}}EP_{\mathcal{C}} = \varepsilon P_{\mathcal{C}}.$$

If $\mathcal{E}$ is an error space, we say $\mathcal{C}$ detects errors from $\mathcal{E}$ (or simply detects $\mathcal{E}$) if the above condition holds for all $E \in \mathcal{E}$. Since the left-hand-side is linear in $E$, the right-hand-side must also be. Hence, $\mathcal{C}$ detects $\mathcal{E}$ if and only if there exists a linear map $\varepsilon \colon \mathcal{E} \to \mathbb{C}$ such that

$$P_{\mathcal{C}}EP_{\mathcal{C}} = \varepsilon(E)P_{\mathcal{C}} \tag{1}$$

for all $E \in \mathcal{E}$. This function $\varepsilon$ is called the *slope* of the code $\mathcal{C}$, and the equation above is called the *error detection condition*. We say $\mathcal{C}$ detects errors of distance $t$ if this condition holds for $\mathcal{E} = \mathcal{V}_t$. Just as in the classical case, we say $\mathcal{C}$ corrects errors of distance $t$ if it detects errors of distance $2t$. If this is the case, there exists a procedure for recovering the original state if an error of distance $\leq t$ occurs.

Although the error detection condition presented above is quite compact, it is somewhat difficult to work with. Here we present an equivalent formulation that is sometimes more useful. Suppose $\{|\psi_1\rangle, \ldots, |\psi_n\rangle\}$ is an orthonormal basis for $\mathcal{C}$. Using the fact that $P_{\mathcal{C}} = \sum_{i=1}^{n} |\psi_i\rangle \langle\psi_i|$, we can compute

$$P_{\mathcal{C}} E P_{\mathcal{C}} = \sum_{1 \leq i,j \leq n} \langle\psi_i| E |\psi_j\rangle |\psi_i\rangle \langle\psi_j| .$$

Hence, the error detection condition holds if and only if

$$\langle\psi_i| E |\psi_j\rangle = \varepsilon(E)\delta_{ij} \tag{2}$$

for all $i$ and $j$.

The following proposition relates error detecting codes in subspaces of $\mathcal{H}$ to the whole space, and we will rely on it heavily in the next section.

**Proposition 5.1.** Let $\mathcal{H} \cong \mathbb{C}^d$ and let $\mathcal{E} \subseteq \mathcal{L}(\mathcal{H})$ be an error space. Let $\mathcal{B}$ be a subspace with orthogonal projection $P_{\mathcal{B}}$, and let $\mathcal{F} = P_{\mathcal{B}} \mathcal{E} P_{\mathcal{B}}$. Note that $\mathcal{F}$ may be interpreted as an error space on $\mathcal{B}$ given by restricting $\mathcal{E}$ to $\mathcal{B}$. Then, if a code $\mathcal{C} \subseteq \mathcal{B}$ detects $\mathcal{F}$, it detects $\mathcal{E}$.

*Proof.* Since $\mathcal{C}$ corrects $\mathcal{F}$, there exists a slope $\varepsilon_0 \colon \mathcal{F} \to \mathbb{C}$ such that

$$P_{\mathcal{C}} F P_{\mathcal{C}} = \varepsilon_0(F) P_{\mathcal{C}} \text{ for all } F \in \mathcal{F}.$$

Since $\mathcal{C} \subseteq \mathcal{B}$, $P_{\mathcal{C}} P_{\mathcal{B}} = P_{\mathcal{B}} P_{\mathcal{C}} = P_{\mathcal{C}}$. Using this fact, we see that for any $E \in \mathcal{E}$

$$\begin{aligned} P_{\mathcal{C}} E P_{\mathcal{C}} &= P_{\mathcal{C}} (P_{\mathcal{B}} E P_{\mathcal{B}}) P_{\mathcal{C}} \\ &= \varepsilon_0(P_{\mathcal{B}} E P_{\mathcal{B}}) P_{\mathcal{C}}. \end{aligned}$$

Hence, if we define $\varepsilon \colon \mathcal{E} \to \mathbb{C}$ by $\varepsilon(E) = \varepsilon_0(P_{\mathcal{B}} E P_{\mathcal{B}})$, we have $P_{\mathcal{C}} E P_{\mathcal{C}} = \varepsilon(E) P_{\mathcal{C}}$ as desired. □

## 6. KLV codes

In [KLV00], Knill, Laflamme, and Viola gave a method for constructing codes that can detect errors from any $\mathcal{V}_t$ in any quantum metric space by transforming the problem to convex geometry.[2] In this section, we'll lay out that construction and go through some examples for Lie type quantum metric spaces. First, we introduce some preliminary results for this construction.

**Definition 6.1.** Let $\{\vec{p}_1, \ldots, \vec{p}_n\}$ be a collection of $n$ points in $\mathbb{R}^d$. Suppose we partition the index set $\{1, 2, \ldots, n\}$ into sets $Y_1, \ldots, Y_r$ such that

$$\bigcap_{j=1}^{r} \operatorname{conv}\{\vec{p}_i \mid i \in Y_j\} \neq \emptyset.$$

If the above condition holds, we call the collection $\{Y_j\}_{j=1}^r$ a *Tverberg partition*, and we call any point $\vec{\varepsilon} \in \bigcap_{j=1}^r \operatorname{conv}\{\vec{p}_i \mid i \in Y_j\}$ a *Tverberg point*. A Tverberg partition is called *maximal* if there does not exist another Tverberg partition of the same points into more parts.

Although we define a Tverberg partition as a particular set partition on the index set, in an abuse of notation we often speak about partitioning the vectors themselves into sets rather than their indices. The following remarkable theorem due to Tverberg gives a lower bound on the number of parts in a maximal Tverberg partition. [Tve66]

**Theorem 6.2** (Tverberg's Theorem)**.** *For any set of $n$ points in $\mathbb{R}^d$, there exists a Tverberg partition with $\lceil n/(d+1) \rceil$ parts.*

KLV's construction goes as follows. Let $\mathcal{H} \cong \mathbb{C}^n$ be a Hilbert space and suppose we wish to find a code $\mathcal{C} \subseteq \mathcal{H}$ that detects the error space $\mathcal{E}$. First, find a subspace $\mathcal{B}$ with projector $P_{\mathcal{B}}$ such that $\mathcal{F} = P_{\mathcal{B}} \mathcal{E} P_{\mathcal{B}}$ is commutative. A greedy argument shows that there exists a such a space space $\mathcal{B}$ of dimension $b \geq \left\lceil \frac{\dim \mathcal{H}}{\dim \mathcal{E} - 1} \right\rceil$.

Next, we look for a code $\mathcal{C} \subseteq \mathcal{B}$ that detects the commutative error space $\mathcal{F} \subseteq \mathcal{L}(\mathcal{B})$. As $\mathcal{F} = \mathcal{F}^*$, $\mathcal{F}$ has a self-adjoint basis $\{I, F_1, \ldots, F_d\}$. Since these errors commute, there is an orthonormal basis $\{|m\rangle\}_{m=1}^b$ for $\mathcal{B}$ in

[2] In their paper, they addressed slightly different structures called quantum graphs, but these can be readily identified with quantum graph metric spaces. Their construction applies equally well to all quantum metric spaces, not just quantum graph metric spaces, so we present the most general case.

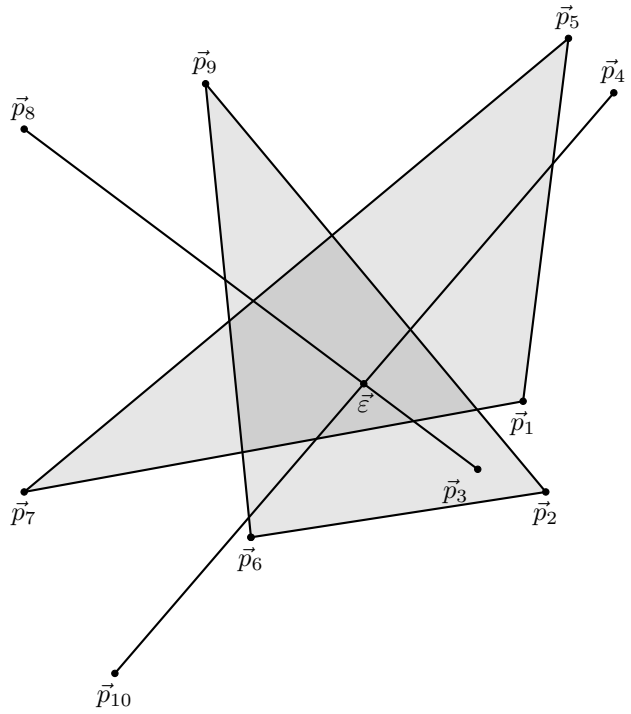



FIGURE 3. A Tverberg partition of 10 points in the $\mathbb{R}^2$ with Tverberg point $\vec{\varepsilon}$ and sets $Y_1 = \{1, 5, 7\}$, $Y_2 = \{2, 6, 9\}$, $Y_3 = \{3, 8\}$, and $Y_4 = \{4, 10\}$.

which all errors are diagonal. To each basis vector $|m\rangle$, we can associate a vector $\vec{\lambda}_m = (\lambda_m^{(1)}, \ldots, \lambda_m^{(d)}) \in \mathbb{R}^d$, where $\lambda_m^{(j)}$ is the eigenvalue of $|m\rangle$ for the matrix $F_j$.

To find a code $\mathcal{C}$ inside $\mathcal{B}$, we find a Tverberg partition $\{Y_j\}_{j=1}^r$ for the set $\{\vec{\lambda}_m\}_{m=1}^b$. Let $\vec{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_d)$ be a Tverberg point for this partition. For each $Y_j$, since $\vec{\varepsilon} \in \operatorname{conv}\{\vec{\lambda}_m \mid m \in Y_j\}$, there exist coefficients $\alpha_{jm} \in [0, 1]$ such that

$$\sum_{m \in Y_j} \alpha_{jm} \vec{\lambda}_m = \vec{\varepsilon}, \qquad \text{and} \qquad \sum_{m \in Y_j} \alpha_{jm} = 1.$$

Consider the vector $|\psi_j\rangle \in \mathcal{B}$ given by

$$|\psi_j\rangle := \sum_{m \in Y_j} \sqrt{\alpha_{jm}} \, |m\rangle \, .$$

We claim that the code $\mathcal{C} = \operatorname{span}_{\mathbb{C}}\{|\psi_j\rangle\}_{j=1}^r$ detects $\mathcal{F}$. By construction, the basis $\{|\psi_j\rangle\}_{j=1}^r$ is orthonormal, so we may check condition (2). Indeed,

$$\begin{aligned}
\langle \psi_j | \, F_l \, | \psi_j \rangle &= \sum_{m \in Y_j} \sum_{m' \in Y_j} \sqrt{\alpha_{m'} \alpha_m} \, \langle m' | \, F_l \, | m \rangle \\
&= \sum_{m \in Y_j} \sum_{m' \in Y_j} \sqrt{\alpha_{m'} \alpha_m} \lambda_m^{(l)} \, \langle m' | m \rangle \\
&= \sum_{m \in Y_j} \alpha_m \lambda_m^{(l)} \\
&= \varepsilon_l .
\end{aligned}$$

And, if $j \neq k$,

$$\begin{aligned}\langle\psi_j| F_l |\psi_k\rangle &= \sum_{m\in Y_j}\sum_{m'\in Y_k} \sqrt{\alpha_m\alpha_{m'}}\,\langle m'| F_l |m\rangle \\ &= \sum_{m\in Y_j}\sum_{m'\in Y_k} \sqrt{\alpha_{m'}\alpha_m}\lambda_m^{(l)}\,\langle m'|m\rangle \\ &= 0.\end{aligned}$$

The last equation holds because $Y_j$ and $Y_k$ are disjoint, so $\langle m'|m\rangle$ for all $m \in Y_j$ and $m' \in Y_k$. Hence the error detection condition holds, so $\mathcal{C}$ detects $\mathcal{F}$. By proposition 5.1, $\mathcal{C}$ detects $\mathcal{E}$, as desired.

Notice that $\dim \mathcal{C}$ is the number of parts in the Tverberg partition of $\{\vec{\lambda}_m\}_{m=1}^n$. Tverberg's theorem asserts that there exists a partition into $\left\lceil \frac{b}{d+1} \right\rceil = \left\lceil \frac{\dim \mathcal{B}}{\dim \mathcal{F}} \right\rceil$ parts. However, this partition is not necessarily maximal, meaning we may be able to find a Tverberg partition with many more parts, which would allow us to construct a larger code.

KLV uses the facts that $\dim \mathcal{F} \leq \dim \mathcal{E}$ and $\dim \mathcal{B} \geq \left\lceil \frac{\dim \mathcal{H}}{\dim \mathcal{E}-1} \right\rceil$ to obtain the bound

$$\dim \mathcal{C} \geq \left\lceil \frac{1}{\dim \mathcal{E}} \left\lceil \frac{\dim \mathcal{H}}{\dim \mathcal{E}-1} \right\rceil \right\rceil = \left\lceil \frac{\dim \mathcal{H}}{\dim \mathcal{E}(\dim \mathcal{E}-1)} \right\rceil. \tag{3}$$

However, this is often deceptively low if $\mathcal{H}$ is a Lie type quantum metric space. In these cases, we can often find a subspace $\mathcal{B}$ much larger than the greedy algorithm suggests. Also, $\dim \mathcal{F}$ is often much smaller than $\dim \mathcal{E}$. And finally, we may be able to find a partition of $\{\vec{\lambda}_m\}_{m=1}^b$ that is significantly larger than the one Tverberg's theorem guarantees. In the following examples, we see all three of these advantages playing out.

**Example 6.3.** Consider the representation $\mathcal{H}_n$ of $\mathfrak{su}(2)$ described in example 4.7. An example is shown below with $n = 8$.

$$\begin{array}{ccccccccccccccccc} |80\rangle & \overset{2\sqrt{2}}{\longleftrightarrow} & |71\rangle & \overset{\sqrt{14}}{\longleftrightarrow} & |62\rangle & \overset{3\sqrt{2}}{\longleftrightarrow} & |53\rangle & \overset{2\sqrt{5}}{\longleftrightarrow} & |44\rangle & \overset{2\sqrt{5}}{\longleftrightarrow} & |35\rangle & \overset{3\sqrt{2}}{\longleftrightarrow} & |26\rangle & \overset{\sqrt{14}}{\longleftrightarrow} & |17\rangle & \overset{2\sqrt{2}}{\longleftrightarrow} & |08\rangle \\ \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft \\ 8 & & 6 & & 4 & & 2 & & 0 & & -2 & & -4 & & -6 & & -8 \end{array}$$

We will construct a distance 1 error-detecting code in this metric space, following [Bum11]. Consider the subspace

$$\mathcal{B} = \text{span}_{\mathbb{C}}\{|n-k,k\rangle \mid k \text{ is even}\},$$

shown in the diagram below by all the boxed vectors.

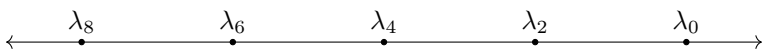
$$\begin{array}{ccccccccccccccccc} \boxed{|80\rangle} & \overset{2\sqrt{2}}{\longleftrightarrow} & |71\rangle & \overset{\sqrt{14}}{\longleftrightarrow} & \boxed{|62\rangle} & \overset{3\sqrt{2}}{\longleftrightarrow} & |53\rangle & \overset{2\sqrt{5}}{\longleftrightarrow} & \boxed{|44\rangle} & \overset{2\sqrt{5}}{\longleftrightarrow} & |35\rangle & \overset{3\sqrt{2}}{\longleftrightarrow} & \boxed{|26\rangle} & \overset{\sqrt{14}}{\longleftrightarrow} & |17\rangle & \overset{2\sqrt{2}}{\longleftrightarrow} & \boxed{|08\rangle} \\ \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft \\ 8 & & 6 & & 4 & & 2 & & 0 & & -2 & & -4 & & -6 & & -8 \end{array}$$

Recall that $\mathcal{E} = \text{span}_{\mathbb{C}}\{I, \rho(E), \rho(F), \rho(H)\}$. Notice that $\rho(E)$ maps each basis vector one space to the right in the above diagram. Hence, it maps each vector in $\mathcal{B}$ to something orthogonal to $\mathcal{B}$, so $P_{\mathcal{B}}\rho(E)P_{\mathcal{B}} = 0$. The same holds for $\rho(F)$, so $P_{\mathcal{B}}\rho(F)P_{\mathcal{B}} = 0$. Hence,

$$\mathcal{F} = P_{\mathcal{B}}\mathcal{E}P_{\mathcal{B}} = \text{span}_{\mathbb{C}}\{I, P_{\mathcal{B}}\rho(H)P_{\mathcal{B}}\}.$$

As $I$ and $P_{\mathcal{B}}\rho(H)P_{\mathcal{B}}$ commute, $\mathcal{F}$ is commutative as desired. Notice that $\dim B = \left\lceil \frac{\dim \mathcal{H}_n}{2} \right\rceil$, whereas the greedy argument employed by KLV only guarantees the existence of such a a subspace of size $\left\lceil \frac{\dim \mathcal{H}_n}{\dim \mathcal{E}-1} \right\rceil = \left\lceil \frac{\dim \mathcal{H}_n}{3} \right\rceil$.

As $\mathcal{F}$ is already diagonal, no change of basis is required. Since the basis $\{I, P_{\mathcal{B}}\rho(H)P_{\mathcal{B}}\}$ only has 1 non-identity error, the associated vector of eigenvalues for each vector $|k, n-k\rangle$ – which we call $\vec{\lambda}_k$ – will only have one coordinate. Since $P_{\mathcal{B}}\rho(H)P_{\mathcal{B}}\,|k, n-k\rangle = n-2k$, we have $\vec{\lambda}_k = (n-2k) \in \mathbb{R}^1$. For $\mathcal{H}_8$, this set of vectors in $\mathbb{R}^1$ is shown below.



To find a code $\mathcal{C}$ inside $\mathcal{B}$, we search for a Tverberg partition of these $\vec{\lambda}_k$'s. One such partition can be constructed by pairing the smallest $\lambda_i$ with the largest, then the second smallest with the second largest, and so on. If there are an odd number of vectors, we can take a singleton containing the last vector in the middle. This yields a Tverberg partition with $\lceil \frac{\dim \mathcal{B}}{2} \rceil$ parts, which is equal to the bound given by Tverberg's theorem. This can be seen to be optimal, since any larger partition would need to contain at least two singletons. However, no Tverberg partition of these points can have more than one singleton, as no two $\lambda_i$'s coincide.

Hence, we have constructed a code of dimension $\lceil \frac{\dim \mathcal{B}}{2} \rceil = \lceil \frac{1}{2} \lceil \frac{\dim \mathcal{H}_n}{2} \rceil \rceil = \lceil \frac{\mathcal{H}_n}{4} \rceil$. A naive application of KLV's bound would tell us there exist codes of dimension $\left\lceil \frac{\dim \mathcal{H}_n}{\dim \mathcal{E}(\dim \mathcal{E}-1)} \right\rceil = \lceil \frac{\dim \mathcal{H}_n}{12} \rceil$. By carefully choosing $\mathcal{B}$, we were able to improve this bound.

**Example 6.4.** Now, consider the representation $\mathcal{H}_n$ of $\mathfrak{su}(3)$. Recall that we can think of our basis vectors $|a_1a_2a_3\rangle$ as being indexed by all elements $(a_1, a_2, a_3) \in \Delta(n, 3)$. Distance $t$ errors take $|a_1a_2a_3\rangle$ to vectors indexed by elements of distance $\leq t$ from $(a_1, a_2, a_3)$ in $\Delta(n, 3)$. In particular, if we take a minimum distance $t + 1$ subset of $\Delta(n, 3)$, then no error $E \in \mathcal{V}_t$ will take an element of this subset to a different element of the subset. Hence, if we take $\mathcal{B}$ to be spanned by these vectors, we have $P_{\mathcal{B}} E P_{\mathcal{B}} = 0$ for all non-diagonal $E \in \mathcal{V}_t$. Hence $P_{\mathcal{B}} \mathcal{V}_t P_{\mathcal{B}}$ will be diagonal.

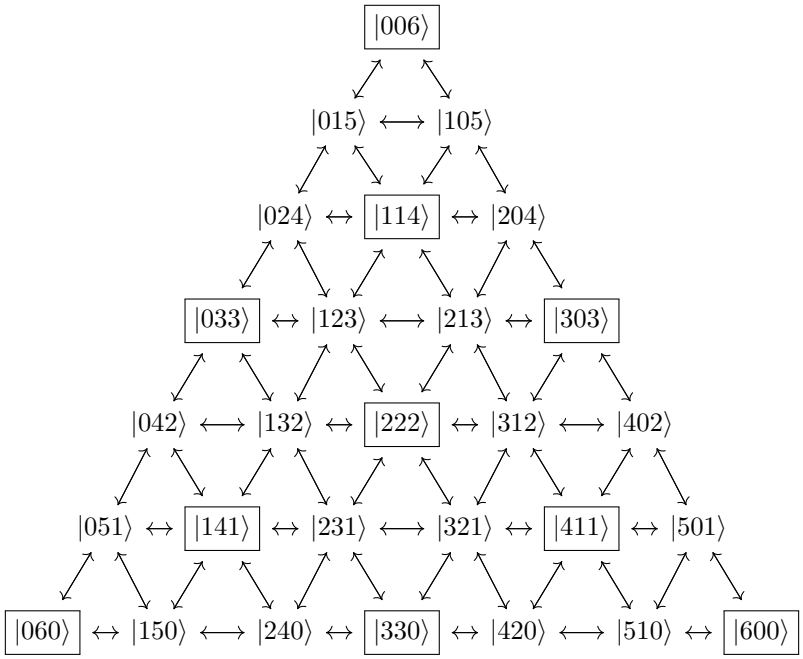



FIGURE 4. Diagram of $\mathcal{H}_6$ for $\mathfrak{su}(3)$. Boxed vectors are the basis vectors for $\mathcal{B}$, and are indexed by elements of $T(6, 3)$.

Recall the minimum distance 2 subset $T(n, 3)$ of $\Delta(n, 3)$ described in example 2.1. Taking $\mathcal{B}$ to be spanned by these basis vectors, we get subspace of dimension $\dim(\mathcal{H}_n)/3 + O(1)$. Explicitly, we have

$$\mathcal{B} = \operatorname{span}\{|a_1a_2a_3\rangle : (a_1, a_2, a_3) \in T(n, 3)\} = \operatorname{span}\{|a_1a_2a_3\rangle : a_1 - a_2 \equiv 0 \pmod 3\}.$$

As argued above, $P_{\mathcal{B}} E P_{\mathcal{B}} = 0$ for all non-diagonal $E \in \mathcal{V}_1$. It's clear from the definition of $\rho$ that $E$ is diagonal if and only if $\rho(E)$ is. It follows that the diagonal elements of $\mathcal{V}_1$ are spanned by $\rho(D_1), \rho(D_2)$, and $\rho(D_3)$, where

$$D_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad D_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad D_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence $P_{\mathcal{B}}\mathcal{V}_1 P_{\mathcal{B}} = \text{span}_{\mathbb{C}}\{P_{\mathcal{B}}\rho(D_1)P_{\mathcal{B}}, P_{\mathcal{B}}\rho(D_2)P_{\mathcal{B}}, P_{\mathcal{B}}\rho(D_3)P_{\mathcal{B}}\}$.

Note that $\rho(D_1)\left|a_1a_2a_3\right\rangle = a_1$, $\rho(D_2)\left|a_1a_2a_3\right\rangle = a_2$, and $\rho(D_3)\left|a_1a_2a_3\right\rangle = a_3$. Hence, the vector of eigenvalues associated with $(D_1, D_2, D_3)$ for $\left|a_1a_2a_3\right\rangle$ is $(a_1, a_2, a_3) \in \Delta(n, 3)$. However, note that when we construct vectors of eigenvalues to look for a Tverberg partition as on page 8, we can disregard the identity error – that is, we find a basis $\{I, F_1, \ldots F_d\}$ for $P_{\mathcal{B}}\mathcal{V}_t P_{\mathcal{B}}$, and only consider eigenvalues for $F_1, \ldots F_d$, meaning our vectors lie in $\mathbb{R}^d$. In this case, this means we only need to consider vectors in $\mathbb{R}^2$ instead of $(a_1, a_2, a_3) \in \Delta(n, 3) \subseteq \mathbb{R}^3$, and this corresponds exactly to the projection of $\Delta(n, 3)$ into $\mathbb{R}^2$ mentioned in example 2.1.

One particularly nice basis for this is given by $\{I, P_{\mathcal{B}}\rho(F_1)P_{\mathcal{B}}, P_{\mathcal{B}}\rho(F_2)P_{\mathcal{B}}\}$, where

$$F_1 = \begin{bmatrix} -1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad F_2 = \begin{bmatrix} -\sqrt{3}/6 & 0 & 0 \\ 0 & -\sqrt{3}/6 & 0 \\ 0 & 0 & \sqrt{3}/3 \end{bmatrix}.$$

The eigenvalues for these operators are

$$\rho(F_1)\left|a_1a_2a_3\right\rangle = \frac{1}{2}(a_2 - a_1)\left|a_1a_2a_3\right\rangle,$$
$$\rho(F_2)\left|a_1a_2a_3\right\rangle = \frac{\sqrt{3}}{6}(2a_3 - a_1 - a_2)\left|a_1a_2a_3\right\rangle.$$

Let $\vec{\lambda}_{a_1a_2a_3} = (\frac{1}{2}(a_2 - a_1), \frac{\sqrt{3}}{6}(2a_3 - a_1 - a_2))$ be the vector of eigenvalues of $\rho(F_1)$ and $\rho(F_2)$ for $\left|a_1a_2a_3\right\rangle$. The set of all such vectors in $\mathbb{R}^2$ for all basis vectors in $\mathcal{B}$ is shown below.

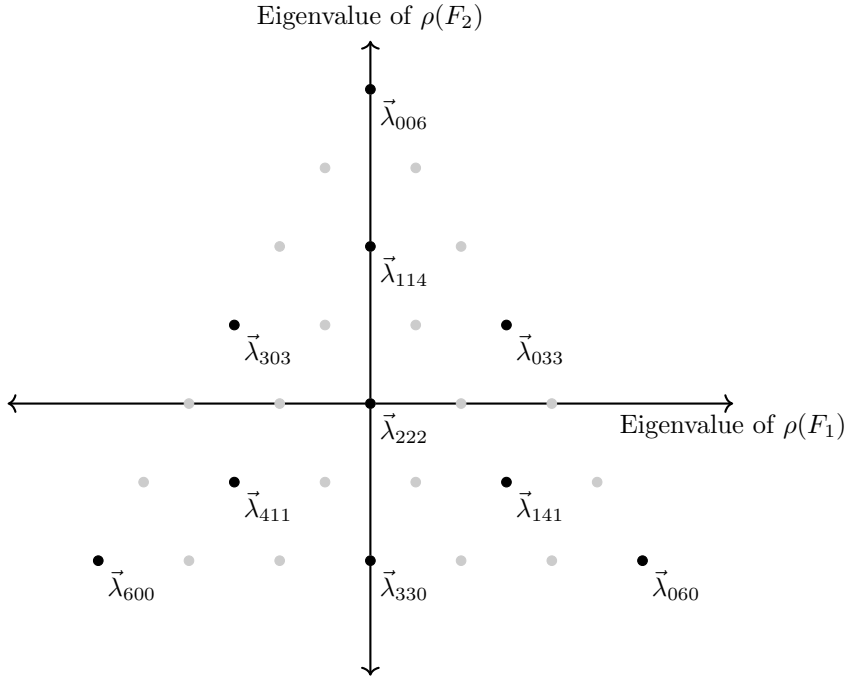



FIGURE 5. Vectors of eigenvalues for $\mathcal{B} \subseteq \mathcal{H}_6$. The eigenvalue vectors for basis vectors of $\mathcal{H}_6$ not in $\mathcal{B}$ are shown in grey.

Now, we need to find a Tverberg partition of these points. If we call the number of points $b$, then Tverberg's theorem asserts that there exists a Tverberg partition with $\lceil b/3 \rceil$ parts. However, since these points are arranged in a very orderly lattice – a projection of $T(n, 3)$ into $\mathbb{R}^2$ – we can show there exists a partition with approximately 4b/9 parts. Such a partition is constructed as follows.

For this construction, we restrict our attention to the case where $n$ is a multiple of 3, so let $m = n/3$.

- Choose $\vec{\varepsilon} = (0, 0) = \vec{\lambda}_{mmm}$ to be the slope of the code. Take the singleton $\{\lambda_{mmm}\}$ in the partition.

- For each $\vec{\lambda}_{a_1a_2a_3}$ such that $a_1, a_2, a_3 \leq 2m$, let $b_1 = 2m - a_1$, $b_2 = 2m - a_2$, and $b_3 = 2m - a_3$. It's clear that $|a_1a_2a_3\rangle \in \mathcal{B}$ implies $|b_1b_2b_3\rangle \in \mathcal{B}$, so $\vec{\lambda}_{b_1b_2b_3}$ is a vector in the set we're partitioning. We then take the pair $\{\vec{\lambda}_{a_1a_2a_3},\ \vec{\lambda}_{b_1b_2b_3}\}$ in the partition. Note that these points are chosen to be on opposite sides of $\vec{\varepsilon}$ – that is,
$$\frac{1}{2}(\vec{\lambda}_{a_1a_2a_3} + \vec{\lambda}_{b_1b_2b_3}) = \vec{\lambda}_{mmm} = \vec{\varepsilon}.$$
So, $\vec{\varepsilon}$ is in the convex hull of this set.
- Finally, suppose $\vec{\lambda}_{a_1a_2a_3}$ does not satisfy $a_1, a_2, a_3 \leq 2m$. If $|a_1a_2a_3\rangle \in \mathcal{B}$, then $|a_2a_3a_1\rangle \in \mathcal{B}$ and $|a_3a_1a_2\rangle \in \mathcal{B}$.[3] We take the 3 element set $\{\vec{\lambda}_{a_1a_2a_3}, \vec{\lambda}_{a_2a_3a_1}, \vec{\lambda}_{a_3a_1a_2}\}$ to be in the partition. Note that
$$\frac{1}{3}(\vec{\lambda}_{a_1a_2a_3} + \vec{\lambda}_{a_2a_3a_1} + \vec{\lambda}_{a_3a_1a_2}) = \vec{\lambda}_{mmm} = \vec{\varepsilon},$$
so $\vec{\varepsilon}$ is in the convex hull of this set.

By construction, this partition is Tverberg. To compute the number of parts, note that the fraction of $\lambda_{a_1a_2a_3}$ such that $a_1, a_2, a_3 \leq 2/3n$ approaches $2/3 + O(1/n)$. Hence, if we employ the construction above, roughly $2/3$ of the points are in sets of size 2, and roughly $1/3$ of them are in sets of size 3. Hence, the total number of parts in the partition is $2b/3(1/2) + b/3(1/3) + O(n) = 4b/9 + O(n)$, as desired.

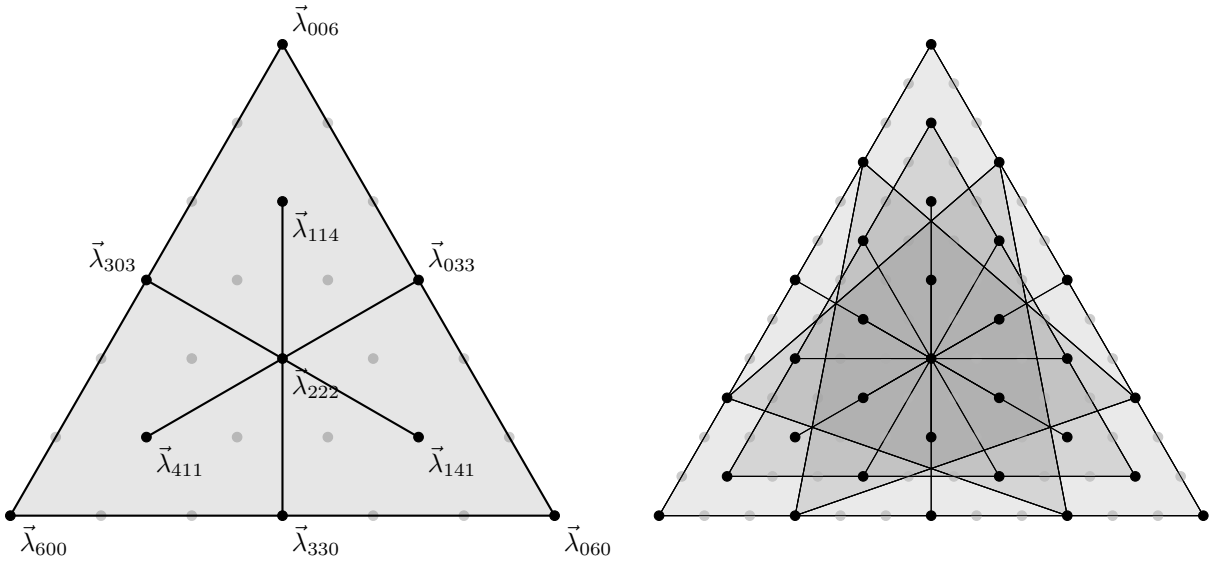



FIGURE 6. LEFT: Tverberg partition for finding a code inside $\mathcal{B} \subseteq \mathcal{H}_6$. The sets in the partition are $\{\vec{\lambda}_{222}\}$, $\{\vec{\lambda}_{033}, \vec{\lambda}_{411}\}$, $\{\vec{\lambda}_{303}, \vec{\lambda}_{141}\}$, $\{\vec{\lambda}_{330}, \vec{\lambda}_{114}\}$, and $\{\vec{\lambda}_{600}, \vec{\lambda}_{060}, \vec{\lambda}_{006}\}$.

RIGHT: Tverberg partition for a larger example, $\mathcal{B} \subseteq \mathcal{H}_{12}$. Notice that this partition consists of 14 parts, whereas Tverberg's theorem only guarantees the existence of a Tverberg partition with 11 parts.

## 7. NON-DIAGONAL CODES

Constructing KLV codes involves choosing a subspace $\mathcal{B}$ such that $P_{\mathcal{B}}\mathcal{V}_t P_{\mathcal{B}}$ is diagonal in some basis. However, we can in some cases find larger codes by picking $\mathcal{B}$ so that $P_{\mathcal{B}}\mathcal{V}_t P_{\mathcal{B}}$ is block diagonal with small

[3]This only holds whenever $n$ is a multiple of 3. Recalling that $|a_1a_2a_3\rangle \in \mathcal{B}$ if and only if $a_1 - a_2 \equiv 0 \pmod 3$ and $a_1 + a_2 + a_3 = n$, we see that if $|a_1a_2a_3\rangle \in \mathcal{B}$,
$$a_2 - a_3 = a_2 - (n - a_1 - a_2) = a_1 + 2a_2 - n \equiv a_1 - a_2 \equiv 0 \pmod 3,$$
so $|a_2a_3a_1\rangle \in \mathcal{B}$. A very similar argument shows $|a_3a_1a_2\rangle \in \mathcal{B}$

blocks. In this case, more work needs to be done to guarantee that the code satisfies the error detection condition.

This technique and the code in example 7.1 were pioneered by Rui Okada, a PhD student who helped tremendously with this project. These results appear in his work in preparation.

**Example 7.1.** Consider the metric space generated by the representation $\mathcal{H}_n$ of $\mathfrak{su}(2)$ of example 4.7, and assume $n = 6m$ for some $m$. Recall that the distance 1 errors in this quantum metric space are spanned $\{\rho(E), \rho(F), \rho(G)\}$, where

$$\begin{aligned}\rho(E)\,|k, n-k\rangle &= \sqrt{k(n-k+1)}\,|k-1, n-k+1\rangle \\ \rho(F)\,|k, n-k\rangle &= \sqrt{(k+1)(n-k)}\,|k+1, n-k-1\rangle \\ \rho(H)\,|k, n-k\rangle &= (n-2k)\,|k, n-k\rangle\end{aligned}$$

In example 6.3 we saw that KLV's construction yields a code of dimension $\left\lceil \frac{\dim \mathcal{H}_n}{4} \right\rceil$. Here we show a non-diagonal code of dimension $\frac{\dim \mathcal{H}_n}{3} + O(1)$.

Choose $\mathcal{B}$ to be spanned by the vectors $|a, b\rangle$ for which $|a - b|$ is congruent to 0 or 2 mod 3, shown boxed in the diagram below.

$$\boxed{|60\rangle} \overset{\sqrt{6}}{\longleftrightarrow} \boxed{|51\rangle} \overset{\sqrt{10}}{\longleftrightarrow} |42\rangle \overset{2\sqrt{3}}{\longleftrightarrow} \boxed{|33\rangle} \overset{2\sqrt{3}}{\longleftrightarrow} |24\rangle \overset{\sqrt{10}}{\longleftrightarrow} \boxed{|15\rangle} \overset{\sqrt{6}}{\longleftrightarrow} \boxed{|06\rangle}$$

$$\begin{array}{ccccccccccc} \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft \\ 6 & & 4 & & 2 & & 0 & & -2 & & -4 \end{array} \qquad -6$$

Recalling that $n = 6m$, notice that $\dim \mathcal{B} = 4m + 1$. For each $j$ from 1 to $m$,

$$|\phi_j\rangle = \alpha_j\,|3m + 3j, 3m - 3j\rangle + \beta_j\,|3m - 3j + 1, 3m + 3j - 1\rangle$$

and

$$|\psi_j\rangle = \alpha_j\,|3m - 3j, 3m + 3j\rangle - \beta_j\,|3m + 3j + 1, 3m - 3j - 1\rangle\,,$$

where

$$\alpha_j = \sqrt{\tfrac{3j+1}{6j+1}}, \qquad \beta_j = \sqrt{\tfrac{3j}{6j+1}}.$$

We claim that the code

$$\mathcal{C} = \mathrm{span}_{\mathbb{C}}\{|3m, 3m\rangle\,, |\phi_1\rangle\,, |\phi_2\rangle\,, \ldots, |\phi_m\rangle\,, |\psi_1\rangle\,, |\psi_2\rangle\,, \ldots, |\psi_m\rangle\}$$

detects errors of distance 1. To see this, one could check equation (2) for each of $\rho(E)$, $\rho(F)$ and $\rho(H)$. Note that $\dim \mathcal{C} = 2m + 1$ is approximately 1/3 the dimension of $\mathcal{H}_n$, as claimed.

It is very likely that a similar technique can be employed to find larger error detecting codes in $\mathfrak{su}(3)$ quantum metric spaces than the one described in section 6. Further work is required to nail down details of such a construction and the determine dimension of the resulting code.

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