On plane partitions and Kasteleyn cokernels

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Abstract

In this report, we discuss the enumeration of plane partitions, a combinatorial object. We discuss two methods that utilize the determinants of specific matrices to solve this enumeration problem, and we investigate the algebraic structure of these matrices. Specifically, we interpret these matrices as homomorphisms, and we are concerned about the structure of the cokernel of the homorphisms. We present conjectures involving the cokernels, and we investigate a specific case of cokernels in an attempt to prove one of the conjectures.

1 Introduction

In this report, we review literature regarding Kasteleyn cokernels. In the paper [5], Greg Kuperberg describes Kasteleyn cokernels and presents conjectures regarding these structures. In order to create these cokernels, we first describe a combinatorial object known as a plane partitions in Section 1. We discuss how a plane partition can be depicted as a lozenge tiling and as a perfect matching. In Section 3 and Section 4, we present two different methods involving matrices that can be used to count the number of plane partitions. We will show how these methods are similar, and we will see that the Kasteleyn-Percus matrices and the Gessel-Viennot matrices are stably equivalent. In Section 5 and Section 6, we construct Carlitz and Jacobi-Trudi matrices using the Gessel-Viennot method. We see that Jacobi-Trudi matrices are a generalization of Carlitz matrices. We can use Carlitz matrices to discuss Kasteleyn cokernels and investigate what is conjectured about them in Section 7. In order to evaluate these matrices further it is useful to introduce q-analogues. We present these in Section 8. We conclude this report by investigating the Smith normal forms of the q-analogues of Carlitz matrices, and by specializing to $a \times b \times 2$ q-Carlitz matrices in order to explore one of the conjectures.

2 Plane Partitions

The heart of this project is grounded in plane partitions. We are interested in investigating plane partitions as they have properties that make them nice to count. These objects are presented by Percy MacMahon in [6]. He concentrates on them in Volume 2 in Section 9 and Section 10. In this book, MacMahon provides a nice formula which demonstrates that the enumeration of plane partitions is round. MacMahon investigates both plane partitions and their q-analogue. Richard Stanley describes q-analogues in Chapter 1 of [9]. A q-analogue of a mathematical object is an object that is in terms of a variable q that becomes the original object when $q = 1$. For example, the q-analogue of an integer n is

$$(n)_q = 1 + q + q^2 + \cdots + q^{n-1} = (1 - q^n)/(1 - q)$$

Similarly, the q-analogue of $n!$ is

$$(n)_q! = (n)_q(n-1)_q \cdots (1)_q$$

and the q-analogue of $\binom{n}{k}$ is

$$\binom{n}{k}_q = (n)_q!/((k)_q!(n-k)_q!)$$

When working with q-analogues, the q-analogue of a binomial coefficient is referred to as a *Gaussian binomial coefficient*. When working with q-analogues, a polynomial is *q-round* if it is a ratio of products of q-integers. It is the case that Gaussian binomial coefficients are both q-round and square free. We will see that the enumeration of plane partitions is round and that the q-enumeration of plane partitions is q-round.

A *plane partition* is an arrangement of unit cubes in an $a \times b \times c$ dimensional rectangular prism which is stable under a gravitational force towards the origin. In other words behind or to the left of every cube is either a cube or another wall. We are concerned in counting the number of plane partition in an $a \times b \times c$ dimensional rectangular prism. There is a natural bijection between plane partitions in an $a \times b \times c$ dimensional rectangular prism and lozenge tilings of a hexagon with side lengths a, b, c, a, b, c covered by equilateral unit triangles. We will refer to such a region as an *$a \times b \times c$ semi-regular hexagon*. A *lozenge tile* is a quadrilateral constructed by attaching two unit equilateral triangles. The bijection arises since a 2-dimensional representation of a plane partition is in fact such a tiling. This bijection transforms the problem of determining the number of $a \times b \times c$ plane partitions into a problem of determining the number of lozenge tilings of an $a \times b \times c$ semi-regular hexagon. In Figure 1 and Figure 2 we display two representations of the same plane partition. The lozenge tiles in Figure 3 are color-coded so that the bijection is more apparent. Typically, the lozenges are all the same color. For the purpose of this paper, we will concentrate on generic plane partitions; however, the enumeration of symmetry classes of plane partitions is also investigated. Stanley explores the symmetry classes in [8]. In this paper he presents 10 symmetry classes that can be constructed through rotation, reflection, and self-complementation. Kuperberg uses determinants and Pfaffians of matrices to express the enumeration of the ten symmetry classes of plane partitions in [3]. Notably he found the formula for the enumeration of the last symmetry class. Even the symmetry classes have round answers which contributes to why plane partitions may be interesting to examine.

In order to determine the number of tilings of the semi-regular hexagonal region, it is useful to utilize another bijection. We can denote the $a \times b \times c$ semi-regular hexagon with a graph where there is a vertex for every equilateral triangle in the region and there is an edge between two vertices if and only if the corresponding equilateral triangles are adjacent in the region. This graph is bipartite. We color a vertex white if it corresponds to an upwards-oriented triangle and we color a vertex black if it corresponds to a downwards-oriented triangle. In the region no two triangles with the same orientation are adjacent, so no two vertices of the same color share an edge, and the graph is bipartite. The graph is planar since we can display it in Figure 2 with no intercepting

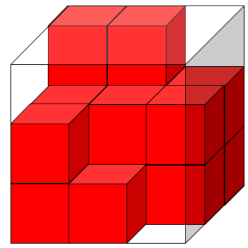

Figure 1: Example of $3 \times 3 \times 3$ plane partition

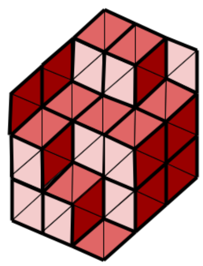

Figure 2: Lozenge tiling of a $3 \times 3 \times 3$ semi-regular hexagon corresponding to the plane partition in Figure 1

edges. Notice that every lozenge tile consists of two equilateral triangles: one oriented upwards and one oriented downwards. Therefore, a lozenge tile corresponds to a pairing of a black vertex with a white vertex. This means that a lozenge tiling of the semi-regular hexagon is in bijection with a perfect matching of the corresponding bipartite graph. A *perfect matching* is a set of edges of a graph where no edges share a vertex and every vertex is covered in the matching. In Figure 3, we display the plane partition from Figure 1 in this manner. The colored edges represent a matching, and they are color-coded so that the color correspond with the color of the lozenge tiles from Figure 2. We can now determine the number of $a \times b \times c$ plane partitions by determining the number of perfect matchings of a bipartite, planar graph.

3 Kasteleyn-Percus Method

In order to determine the number of perfect matchings of a bipartite planar graph, G, we will need to consider the graph's *alternating adjacency matrix* denoted by A. To construct this matrix, we label each of the vertices in G. For convenience, since G is bipartite, we can 2-color G such that no two vertices of the same color share an edge. Say that under the 2-coloring there are m vertices of color 0 and n vertices of color 1. We label the vertices of color 0 with the integers $\{1,2,\ldots,m\}$, and we label the vertices of color 1 with the integers $\{m+1,m+2,\ldots,m+n\}$. We let $A_{i,j}$ be the number of edges from vertex i to vertex j minus the number of edges from vertex j to vertex i. It follows that A is an $(m+n)\times(m+n)$ matrix. Since G is bipartite, if both i and j are in $\{1,2,\ldots,m\}$ or both i and j are in $\{m+1,m+2,\ldots,m+n\}$ then $A_{i,j}=0$. Thus the adjacency

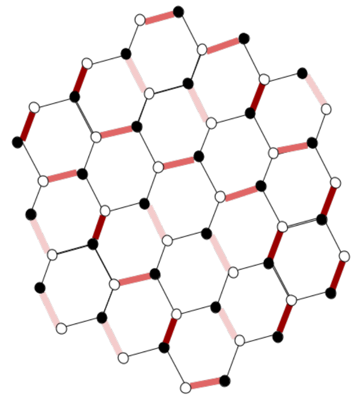

Figure 3: Perfect matching of the bipartite, planar graph corresponding to a $3 \times 3 \times 3$ plane partition

matrix is of the form,

$$A = \left(\begin{array}{c|c} 0 & M \\ \hline -M & 0 \end{array} \right)$$

where M is the *bipartite adjacency matrix* of G. We will concentrate on M which is $m \times n$ dimensional. If $m = n$, then the number of vertices of color 0 is the same as the number of vertices of color 1. In this case, M is a square matrix, and it is possible that G has a perfect matching.

Recall that the determinant of an $n \times n$ matrix $M = (m_{i,j})$ is

$$\det M = \sum_{\sigma} sgn(\sigma) \prod_{i=1}^{n} m_{i,\sigma(i)}$$

where σ is a permutation of n objects. We also consider the permanent of an $n \times n$ matrix which is

$$\text{perm } M = \sum_{\sigma} \prod_{i=1}^{n} m_{i,\sigma(i)}$$

Note that if the entries of M are only zeros and ones then the permanent of M is the number of permutations,σ, of n objects such that $m_{i,\sigma(i)} = 1$ for all $1 \leq i \leq n$. If M is a bipartite adjacency matrix, then such a permutation is a perfect matching. The permanent of a bipartite adjacency matrix is the number of perfect matchings in the corresponding graph. Since we know more about the determinant of a matrix than the permanent, if we can find a connection between the determinant and the permanent, we can efficiently calculate the number of $a \times b \times c$ plane partitions.

Theorem 3.1 (Kasteleyn-Percus Method)**.** *Let G be a simple, planar, bipartite graph and let M be the bipartite adjacency matrix of G. The graph G admits a sign decoration, M' such that all terms in* $\det M$*' have the same sign, where M' is the bipartite adjacency matrix of G with the sign decoration.*

The Kasteleyn-Percus method was created by mathematicians Pieter Kasteleyn and Jerome Percus. They discuss this process in [2] and in [7], respectively. Theorem 3.1 appears in [5]. This is also referred to as the *permanent-determinant method*. Note that if all terms in $\det M'$ are the same sign, then $|\det M'| = \text{perm}\, M$ as desired. The sign decoration of G gives a face of G an odd number of negative signs if and only if the face has $4k$ sides for some integer k. The graph G is said to be *Kasteleyn-flat* when placed under such a sign orientation and the corresponding bipartite adjacency matrix M' is called a *Kasteleyn-Percus matrix*. A similar technique can be applied to general planar graphs and is known as the *Hafnian-Pfaffian method*. This method uses the Pfaffian of a matrix to express the number of perfect matching, and orients the graph such that the the Hafnian of the oriented graph is the same as the Pfaffian of the original graph. This method is used in [3]. For the scope of this report, we will be concentrating on bipartite, planar graphs and the Kasteleyn-Percus method.

By using the Kasteleyn-Percus method and the bijections in Section 2, we can construct the clean formula for the number of $a \times b \times c$ plane partitions that MacMahon found, $PP(a,b,c)$.

$$PP(a,b,c) = \frac{H(a+b+c)H(a)H(b)H(c)}{H(a+b)H(a+c)H(b+c)}$$

Where $H(n) = (n-1)!(n-2)!\dots(1)!$ is the hyper factorial. This formula is noteworthy since it is *round* or the product of small factors. This property inspires further research into plane partitions as it demonstrates that something interesting arises in the background of the problem.

4 Gessel-Viennot Method

It is common to use determinants of specific matrices to count combinatorial objects. The Gessel-Viennot method works similarly. This is presented in [1]. Let G be a directed, plane graph with n sources and n sinks. Let the edges at each vertex be segregated so that there are no four edges alternating in, out, in, out. The Gessel-Viennot method constructs an $n \times n$ matrix, M, such that $\det M$ is the number of disjoint, directed paths in G from the sources to the sinks. In the Gessel-Viennot matrix, $GV(G)$, the entry $GV(G)_{i,j}$ is the number of directed paths from the source i to the sink j.

There is a connection between the Gessel-Viennot method and the Kasteleyn-Percus method. This is a result of Kuperberg in [4]. Suppose that G is a graph on which we can apply the Gessel-Viennot method. We want to produce a modified graph G' that we can apply the Kasteleyn-Percus method to. Let G have a transit vertex p. In G' we split the transit vertex into two vertices q and r where q is a sinky and r is source and there is an edge directed from r to q. Figure 4 shows a vertex being split.

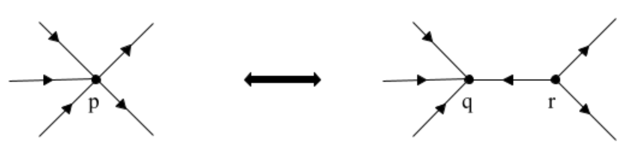

Figure 4: Depiction of vertex splitting

It follows that G' is planar since G is planar, and G' is bipartite since all of the vertices are sinks or sources. There cannot be an edge from one source to another source or one sink to another sink, so if we color all of the sources color 0 and all of the sinks color 1, it is clear that G' is bipartite. For the purpose of applying the Kasteleyn-Percus method, G' need not be directed.

There is a bijection between the number of disjoint, directed paths in G from the sources to the sinks and the number of perfect matchings in G'. Take a set of of disjoint, directed paths in G and transform G into G' by splitting the transit vertices. If there is an edge between two vertices in G in the set of paths then let the two vertices be matched in G'. If a vertex in G is not contained in any of the edges in the set of paths, then the vertex must be a transit vertex. Thus, in G' the vertex is split into two separate vertices that are connected. Let these two new vertices be matched in G'. This does result in a perfect matching in G'. Therefore, the Gessel-Viennot matrix of G, $GV(G)$, and the Kasteleyn-Percus matrix of G', $KP(G')$, have determinants which enumerate the same thing.

It is actually the case that $GV(G)$ and $KP(G')$ share several properties as they are stably equivalent. Let R be a commutative ring with unity, and let M be a matrix over R. There are three types of equivalences that we consider: general row operations, general column operations, and stabilization. These are respectively:

$$M \mapsto AM, M \mapsto MA, M \mapsto \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & M \end{array}\right)$$

where A is invertible. Any matrix M' which can be transformed to M under these operation is a *stably equivalent form* of M We define a pivot operation on square matrices M to be:

$$\left(\begin{array}{c|c} M & v \\ \hline w & 1 \end{array}\right) \mapsto M - (v \otimes w)$$

This operation does not change the determinant of the matrix, and matrices are stably equivalent under the pivot operation. It can be shown that $GV(G)$ can be obtained by applying pivot operations to $KP(G')$. Thus, $GV(G)$ is a stably equivalent from of $KP(G')$. It can be more convenient to work with $GV(G)$ than with $KP(G')$.

5 Carlitz Matrices

We want to investigate the structure behind the Kasteleyn-Percus matrices used to enumerate $a \times b \times c$ plane partitions. As shown in Section 3, this can be performed by examining the structure of Gessel-Viennot matrices. We define *Carlitz matrices* to be the Gessel-Vienot matrices that describe $a \times b \times c$ plane partitions denoted as $C(a,b,c)$. These matrices are mentioned by Kuperberg in [4]. In order determine the entries of Carlitz matrices we transform the bipartite, planar graph whose number of perfect matchings is the number of $a \times b \times c$ plane partions into a graph on which we can apply the Gessel-Viennot method. This is done by performing the opposite of a vertex splitting. We contract the edges in the graph which are parallel, and we direct the edges such that vertical edges run from south to north and horizontal edges run from west to east. By our chosen convention we contract the edges so that there are c sources and c sinks. This transformation is depicted in Figure 5.

The Gessel-Viennot matrix of this graph will be $c \times c$ dimensional. We need only determine the number of northeastern lattice paths from source i to sink j. This is $\binom{a+b}{b+i-j}$. Therefore, the Carlitz matrix, $C(a,b,c)$, has entries $C_{i,j} = \binom{a+b}{b+i-j}$. Although, we choose to contract so that the matrix is

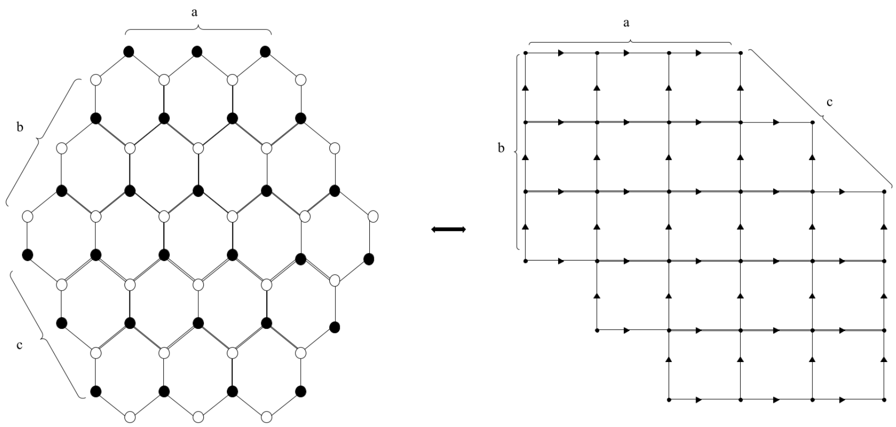

Figure 5: Correspondence between the bipartite, planar graph for $a \times b \times c$ plane partition and the graph modified for Gessel-Viennot method

$c \times c$ dimensional, we can contract in any of the three dimensions, and all three resulting matrices are stably equivalent. From now on we will arrange a, b, c such that $a \geq b \geq c$. This allows us to focus on the matrix that is stably equivalent to the Kasteleyn-Percus matrix used to enumerate $a \times b \times c$ plane partitions which has the least dimension.

6 Jacobi-Trudi Matrices

In [5], Kuperberg introduces Jacobi-Trudi matrices in the context of Kastelyn cokernels. While we have focused on lozenge tilings in a semi-regular hexagon, we can also consider other regions. For example, we can consider lozenge tilings over a trapezoidal region. Consider a trapezoid with height a and bases with length c and $a+c$ that is covered by unit equilateral triangles. The number of tilings over this region is zero as there are a more upwards-oriented triangles than downwards-oriented triangles. We can correct for this by appending a upwards-oriented triangles on top of the larger base of the trapezoidal region. We call the appended triangles *teeth*, and we use the set $T = \{t_1, t_2, \ldots, t_a\}$ to denote the location of the teeth. For convention we index the location of teeth starting at 0. $T \subset \{0, 1, ..., a+c-1\}$ and $|T| = a$. These regions are referred to as *trapezoids with teeth.* Note that instead of appending a upwards-oriented triangles, it is also valid to remove a downwards-oriented triangles on the largest row. This also results in a region with an equinumerous number of triangles with both orientations. We refer to the missing triangles as *gaps*, and call this region *trapezoids with gaps.* If a trapezoid with teeth and a trapezoid with gaps have their teeth or gaps at corresponding locations, they have the same number of tilings. This is true since in a tiling of a trapezoid with teeth, all teeth must be covered by a lozenge tile that also covers an adjacent downwards-oriented triangle. Thus, for any perfect tiling of a trapezoid with gaps, we can attach lozenge tile to the gaps to have a tiling of a trapezoid with teeth, and for any perfect tiling of a trapezoid with teeth, we can remove the tiles that cover the teeth in order to have a tiling of a trapezoid with gaps. The tilings are in bijection so they number of tilings of both regions is indeed the same. Depending on the case, it can be helpful to think about both regions. These are depicted in Figure 6.

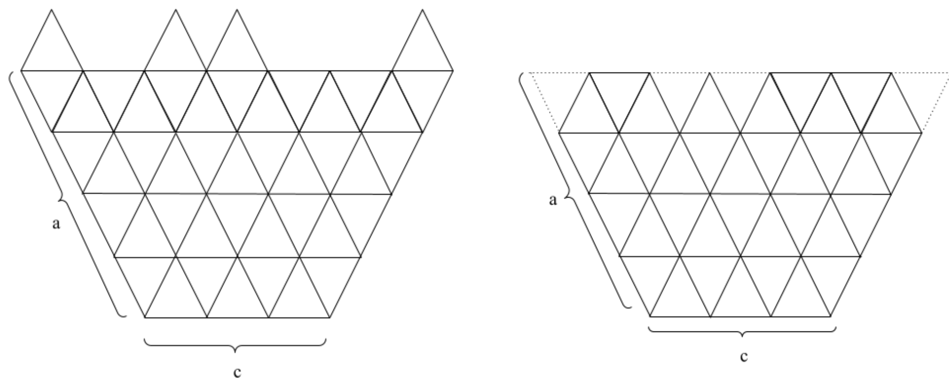

Figure 6: Trapezoid with Teeth and Trapeozoid with gaps where $a = 4$,$c = 3$, and $T = \{0, 2, 3, 6\}$

We want to construct a matrix whose determinant is the number of tilings of an $a \times c$ trapezoid with teeth located at $t_1, t_2, \ldots, t_a$. Similarly to the semi-regular hexagon case, we will use the Gessel-Viennot method to construct these matrices. The trapezoidal region is not as symmetric as the semi-regular hexagons, so the direction of edge we choose to contract matters. If we contract on the backwards-slanted edges in the bipartite, planar graph coresponding to trapezoids with teeth, then the Gessel-Viennot matrix produced is referred to as a *Jacobi-Trudi matrix*. When we perform the contraction we also reflect over the horizotal axis. Thus, the sources are the teeth. This is shown in Figure 7. If we contract on the vertical edges in the bipartite, planar graph corresponding to trapezoids with gaps, then the Gessel-Viennot matrix produced is referred to as a *dual Jacobi-Trudi matrix*. In this graph the sinks correspond to the triangles on the top row where there are not gaps. This is depicted in Figure 8. Note that in Figure 8 we depict dashed edges and vertices to represent what was removed by the gaps. With these matrices we determined equations for Jacobi-Trudi matrices, $JT(a,c,T)$, and for dual Jacobi-Trudi matrices, $dJT(a,c,T)$. The Jacobi-Trudi matrices are $a \times a$ dimension and have entries $JT_{i,j} = \binom{a+c-1-t_i}{c-t_i+j-1}$. Let $S = T^c$ have elements $s_1, s_2, \ldots, s_c$. The dual Jacobi-Trudi matrices are $c \times c$ dimensional and have entries $dJT_{i,j} = \binom{a}{a+i-1-s_j}$

The Jacobi-Trudi matrices are actually a generalization of Carlitz matrices. If the trapezoid with teeth has base lengths $a+b+c$ and c and the set of teeth are $\{0, 1, \ldots, a-1, a+b+1, a+b+2, \ldots, a+b+c\}$, then the Jacobi-Trudi matrices corresponding to this region are stably equivalent to the Carlitz matrix $C(a,b,c)$.

7 Kasteleyn Cokernels

We have now described how to construct Carlitz, Jacobi-Trudi, and dual Jacobi-Trudi matrices. We were motivated to construct these matrices in order to enumerate plane partitions or lozenge tilings, and we have found that this enumeration results in round answer. This inspired us to investigate what may be occurring in the background to result in such a “nice” answer. A square matrix with dimensions $n \times n$ over a ring R can also be interpreted as a homomorphism from R^n to

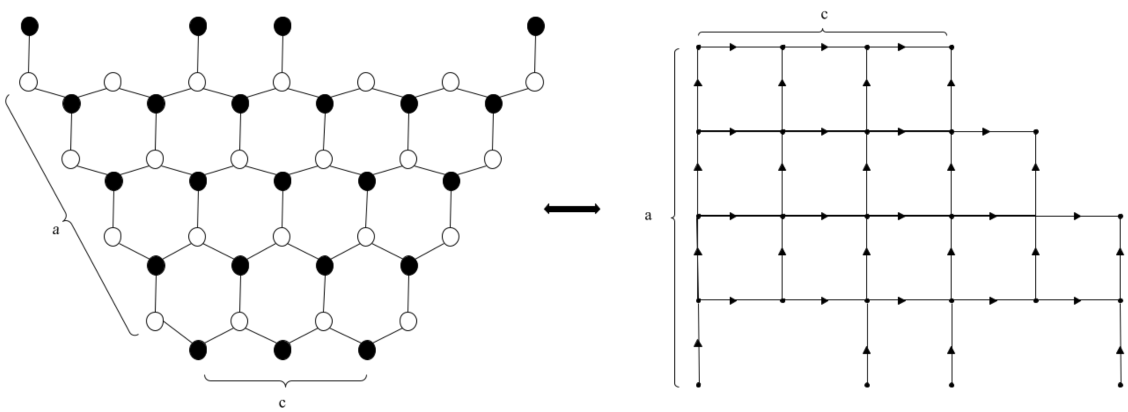

Figure 7: Correspondence between the bipartite, planar graph for trapezoids with teeth and the graph modified for Gessel-Viennot method

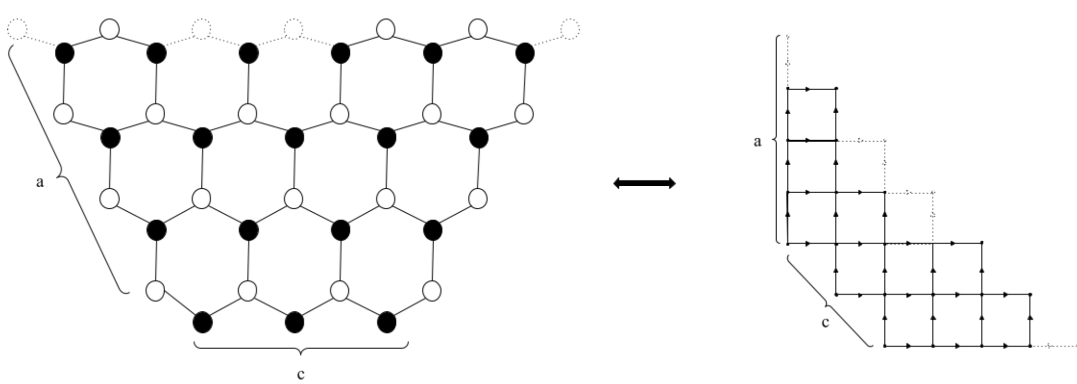

Figure 8: Correspondence between the bipartite, planar graph for trapezoids with gaps and the graph modified for Gessel-Viennot method

itself. This interpretation has a cokernel

$$\text{coker } M = R^n/\text{im}M$$

The cokernel is also the free R-module with n generators whose relations are given by M. If M is the stably equivalent form of M' then the cokernels of M and M' are the same. The number of elements in the cokernel is the determinant of M up to unit factors. We are interested in investigating the structure of the cokernels of Kasteleyn-Percus matrices used to enumerate plane partitions. We refer to these cokernels as *Kasteleyn cokernels*, and they are the subject of [5].

If R is a PID, then the cokernel of M can be described by the Smith normal form of M, $Sm(M)$.

Theorem 7.1. *If M is a $k \times n$ matrix over some PID R, then there exists invertible matrices A and B such that*

$$Sm(M) = AMB$$

is diagonal and $Sm(M)_{i,i}$ divides $Sm(M)_{i+1,i+1}$

As the Carlitz matrices are integer matrices, they all admit Smith normal forms, and we can quickly examine the cokernel of the Carlitz matrices. Results involving Kasteleyn cokernels are found in [4]. It follows by the Gessel-Viennot method that if one of a,b,c is 1, then the cokernel of $C(a,b,c)$ is cyclic. Kasteleyn cokernels need not be cyclic. For example, $\text{coker } C(2,2,2) \cong \mathbb{Z}/2 \times \mathbb{Z}/10$. We used computer coding to investigate the structure of Kasteleyn cokernels. After determining the Smith normal form of $C(a,b,c)$ where $2 \leq a \leq 15$, $2 \leq b \leq 50$, $2 \leq c \leq 50$ we found that $C(4,2,2), C(2,4,2), C(2,2,4)$ were the only Carlitz matrices with a cyclic cokernel. We conjecture that these are the only 3 Carlitz matrices such that $\min(a,b,c) > 1$ with cyclic cokernels.

We are interested in the number of non unit entries in the Smith normal form of a Carlitz matrix. This number corresponds to the rank of the cokernel of $C(a,b,c)$. By the Gessel-Viennot method, we know that the rank of the cokernel of $C(a,b,c) \leq \min(a,b,c)$. We ran computer code to determine the rank of $C(a,b,c)$. We let $2 \leq a \leq 50$, $2 \leq b \leq 20$, and $2 \leq 20$. We found that for the majority of these matrices the cokernel of $C(a,b,c)$ is indeed $\min(a,b,c)$. Moreover, the difference between the rank of $C(a,b,c)$ and $\min(a,b,c)$ was at most 3. When investigating Carlitz matrices of the from $C(a,a,a)$ with $1 \leq a \leq 40$, we noticed that the rank of $C(a,a,a)$ is a when a is a power of 2. We conjecture that $C(a,a,a) = a$ if and only if a is a power of 2.

8 q-analogue of Plane Partitions

While investigating the structure of the Kasteleyn cokernels we desired to understand what else may be occurring that we do not notice when working over the integers. In order to further investigate these cokernels, we want to investigate the q-analogues of Carlitz matrices. We construct a q-analogue for plane partitions by assigning a plane partition the weight of q^n if the plane partition has exactly n unit cubes. We assign weights to the edges in the bipartite planar graph that describes $a \times b \times c$ plane partitions, so that the product of the weights of a perfect matching is q^n if and only if the corresponding plane partition has n unit cubes. This weighting is done by giving all of the vertical and backwards-slanted edges a weight of 1, and by giving the bottom rightmost forwards-slanted edges a weight of 1. The weights of the other forwards-slanted edges is q times the weight of the forwards-slanted edges to the lower-right of it. We use the Gessel-Viennot

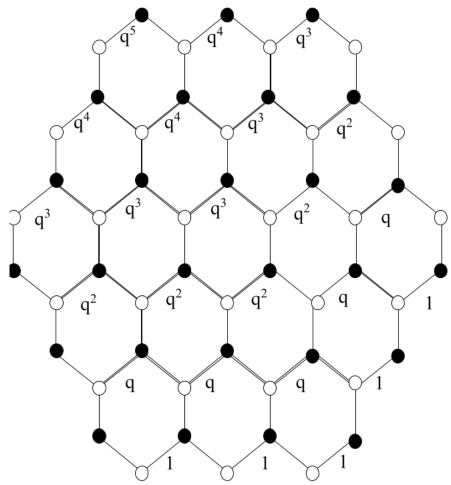

Figure 9: Example of the q-weighting on the bipartite, planar graph for $3 \times 3 \times 3$ plane partitions

method to construct the q-analogue of the Carlitz matrices, and we account for extra factors of q under the weighting. The entries of the q-Carlitz matrices, denoted by $C(a,b,c;q)$ must include extra factors of q. We are going to work in a ring where q is a unit, so for convenience we multiply all entries by a power of q. We have that

$$C(a,b,c;q)_{i,j} = q^{(c-1-i)j} \binom{a}{b+i-j}_q$$

Note that $C(a,b,c;1) = C(a,b,c)$ as desired. It should also be noted that MacMahon also considered q-enumeration while working with plane partitions in [6]. He found the formula for the q-enumeration without utilizing neither the Kasteleyn-Percus method nor the Gessel-Viennot method.

We can apply a similar weighting to the graphs of trapezoids with teeth in order to construct the q-analogue for Jacobi-Trudi and dual Jacobi-Trudi matrices. The desired weighting has the same construction. We can then determine q-analogue of the Jacobi-Trudi matrices, denoted as $JT(a,c,T;q)$, and the q-analogue of the dual Jacobi-Trudi matrices, denoted as $dJT(a,c,T;q)$. These were determined by the author. As in the case of the q-Carlitz matrices, the main difference is the factor of q each entry is multiplied by. The entries of the q-Jacobi-Trudi matrices are

$$JT_{i,j} = q_1(t_i,j) \binom{a+c-1-t_i}{c-i_1+j-1}_q$$

where

$$q_1(t_i,j) = \begin{cases} (j-1)(c-t_i) + \sum_{k=1}^{j-2} k & t_i \leq c-1 \\ \sum_{k=1}^{c+j-t_i-2} k & t_i > c-1 \end{cases}$$

The entries of the q-dual Jacobi-Trudi matrices are

$$dJT_{i,j} = q_2(i,s_j) \binom{a}{a+i-1-s_j}_q$$

where

$$q_2(i,s_j) = \begin{cases} \sum_{k=1}^{c-i} k(s_j - c)(c-i) + & s_j \geq c \\ \sum_{k=1}^{s_j - i} k & s_j < c \end{cases}$$

9 Smith Normal Forms of q-Carlitz matrices

We will now investigate the cokernels of q-Carlitz matrices over the ring $\mathbb{Z}[q,q^{-1}]$. It is conjectured in [5] that these matrices not only admit a Smith normal form over $\mathbb{Z}[q,q^{-1}]$ with q-round answers but also that the entries are square-free. This ring is a UFD, but it is not a PID. Matrices with entries over a UFD need not admit a Smith normal form, and most matrices over $\mathbb{Z}[q,q^{-1}]$ do not have a Smith normal form. Since $\mathbb{Q}[q]$ is a PID, we can determine the Smith normal form of $C(a,b,c;q)$ over $\mathbb{Q}[q]$ to determine the Smith normal form of $C(a,b,c;q)$ over $\mathbb{Z}[q,q^{-1}]$ if it exists.

We further conjecture that if a $C(a,b,c;q)$ matrix admits a Smith normal form then the number of nonzero non unit entries of the Smith normal form of $C(a,b,c;q)$ is $\min(a,b,c)$. This conjecture would explain why the rank of cokernel of Carlitz matrix, $C(a,b,c)$ is often $\min(a,b,c)$. For example, the Smith normal form of $C(3,3,3)$ has diagonal entries 1,7, and 140. This has one unit entry, but the Smith normal form of $C(3,3,3;q)$ has diagonal entries:

$$q^2-q+1, (q^2-q+1)(q^6+q^5+q^4+q^3+q^2+q+1),$$

and

$$q(q+1)(q^2-q+1)(q^4+1)(q^4+q^3+q^2+q+1)(q^6+q^5+q^4+q^3+q^2+q+1)$$

The q-Carlitz matrix does not have any non unit entries on the diagonal. This conjecture also supports that the cokernels of $C(a,b,c)$ are not cyclic if $\min(a,b,c) > 1$ with the exception of (2,2,4),(2,4,2) and (4,2,2). The cokernel of $C(a,b,c)$ is not cyclic if two of the entries in the diagonal of the Smith normal form of $C(a,b,c)$ have a non unit greatest common divisor. If the conjecture holds and $\min(a,b,c) > 1$, then the Smith normal form of $C(a,b,c;q)$ has 2 non unit entries on the diagonal. By the construction of a Smith normal form, one of these entries divides the other, so these entries have a non unit greatest common divisor.

10 Cokernels of $C(a,b,2;q)$ over $\mathbb{Z}[q,q^{-1}]$

We would like to make progress on the conjecture that $C(a,b,c;q)$ does in fact admit a Smith normal form over $\mathbb{Z}[q,q^{-1}]$. If any of a,b,c is 1 then by the Gessel-Viennot method, $C(a,b,c;q)$ trivially has a Smith normal form. We now suppose that $a,b,c \geq 2$, and we specialize to the case where $c = 2$. In this case $C(a,b,2;q)$ is

$$\begin{bmatrix} \binom{a+b}{b}_q & \binom{a+b}{b+1}_q \\ q\binom{a+b}{b-1}_q & \binom{a+b}{b}_q \end{bmatrix}$$

which is stably equivalent to

$$C'(a,b,2;q) = \begin{bmatrix} \binom{a+b}{b}_q & q\binom{a+b}{b+1}_q \\ \binom{a+b}{b-1}_q & \binom{a+b}{b}_q \end{bmatrix}$$

We could quickly convert this matrix into its Smith normal form if there is one entry the divides the other three entries. The likely candidate for such an entry is $\binom{a}{b-1}_q$. We want to investigate when $\binom{n}{k}_q$ divides $\binom{n}{k+1}_q$ and $\binom{n}{k=2}_q$. We can expand these Gaussian binomial coefficients to get that

$$\binom{n}{k+1}_q = \binom{n}{k}_q \frac{(n-k)_q}{(k+1)_q}$$

$$\binom{n}{k+2}_q = \binom{n}{k}_q \frac{(n-k)_q(n-(k+1))_q}{(k+2)_q(k+1)_q}$$

By the first equation we see that:

(1) if $(k+1)_q$ divides $(n-k)_q$, then $\binom{n}{k}_q$ divides $\binom{n}{k+1}_q$.

By the second equation equation we get that:

(2) if $(k+2)_q(k+1)_q$ divides $(n-k)_q(n-(k+1))_q$, then $\binom{n}{k}_q$ divides $\binom{n}{k+2}_q$.

It is known that $(d)_q$ divides $(n)_q$ if and only if $d|n$, so (1) is satisfied if $n \equiv k \mod k+1$. We have that (2) is satisfied if:

a. if $n \equiv k \mod (k+1)(k+2)$,

b. if $n \equiv k+1 \mod (k+1)(k+2)$,

c. if $n \equiv k \mod (k+1)$ and $n \equiv k+1 \mod (k+1)$ which by the Chinese remainder theorem is satisfied when $n \equiv -1 \mod (k+1)(k+2)$,

d. if $n \equiv k+1 \mod (k+1)$ and $n \equiv k+2 \mod (k+1)$ which by the Chinese remainder theorem is satisfied when $n \equiv 2k+2 \mod (k+1)(k+2)$.

In order to satisfy both (1) and (2) we can have that $n \equiv k \mod (k+1)(k+2)$ or that $n \equiv -1 \mod (k+1)(k+2)$.

Suppose that $n \equiv k$ or $-1 \mod (k+1)(k+2)$. Take $b-1=k$ and $a+b=n$, In other words $b=k+1$ and $a \equiv -1$ or $-k-2 \mod (k+1)(k+2)$. Then for such an a and b, $C'(a,b,2)_{2,1}$ divides all entries of $C'(a,b,2;q)$, and these $C(a,b,c;q)$ admit a Smith normal form over $\mathbb{Z}[q,q^{-1}]$.

These are not the only cases where q-Carlitz matrices admit a Smith normal form over this ring. Consider $C(2,2,2;q)$. We can convert this matrix into its Smith normal form using invertible row and column operations.

$$C'(2,2,2;q) = \begin{bmatrix} q^4+q^3+2q^2+q+1 & q^4+q^3+q^2+q \\ q^3+q^2+q+1 & q^4+q^3+2q^2+q+1 \end{bmatrix}$$

$$\mapsto \begin{bmatrix} q^2+1 & -q^5-q^3 \\ q^3+q^2+q+1 & q^4+q^3+2q^2+q+1 \end{bmatrix}$$

$$\mapsto \begin{bmatrix} q^2+1 & 0 \\ q^3+q^2+q+1 & -q^6-q^5+2q^2+q+1 \end{bmatrix}$$

$$\mapsto \begin{bmatrix} q^2+1 & 0 \\ 0 & (q^2+1)(q^4+q^3+q^2+q+1) \end{bmatrix}$$

Notice that during this transformation, we produce the entry q^2+1 which is the greatest common divisor of $q^4+q^3+2q^2+q+1$ and q^3+q^2+q+1. The author believes that this may be importing in proving the conjecture. We conjecture that the equation

$$\binom{n}{k}_q x + \binom{n}{k+1}_q y = \gcd(\binom{n}{k}_q, \binom{n}{k+1}_q)$$

has solutions in $\mathbb{Z}[q,q^{-1}]$. This equation is the *Bézout equation* and always has solutions in a PID. This equation is used when converting a matrix into its Smith normal form over a PID, and if this conjecture is true, it could allow us to construct an algorithm to convert q-Carlitz matrices into their Smith normal form. We have found that

$$\gcd(\binom{n}{k}_q, \binom{n}{k+1}_q) = \frac{(\gcd(n+1,k+1))_q}{(k+1)_q}\binom{n}{k}_q$$

This can be used to generate solution to the equation for specific cases. For example, if $(k+1)|n$ and $\gcd(k+1,n+1)=1$, then $(\frac{-q(n-k-1)_q}{(k+1)_q}, 1)$ solves the equation for $k<n/2$ and $(1, \frac{-q(n-k-1)_q}{(k+1)_q})$ solves the equation for $k>n/2$.

11 Concluding Remarks

There is still a lot of progress to be made in proving these conjectures. We have proven that infinitely many q-Carlitz matrices admit a Smith normal form over $\mathbb{Z}[q,q^{-1}]$, and we are hopeful that an algorithm can be created to convert $C(a,b,2;q)$ into its Smith normal form. We are also interested in investigating these conjectures in connection with Jacobi-Trudi matrices. We have constructed q-specializations of a Jacobi-Trudi matrix, and it is conjectured that these matrices admit a Smith normal form as well. We wish to look into the rank of these matrices.

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