Quantum Error Detection in a General Metric Space Setting

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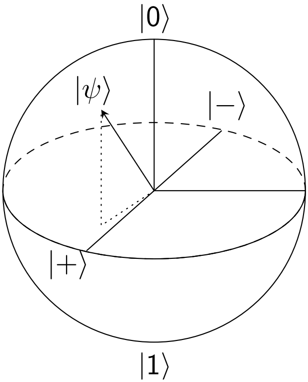
Contents

Where We Live



The state of a single qubit can be written as

$$|\psi_{\text{qubit}}\rangle = a\,|0\rangle + b\,|1\rangle = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Its state space is

$$|\psi_{\text{qubit}}\rangle \in \mathcal{H}_2 \cong \mathbb{C}^2.$$

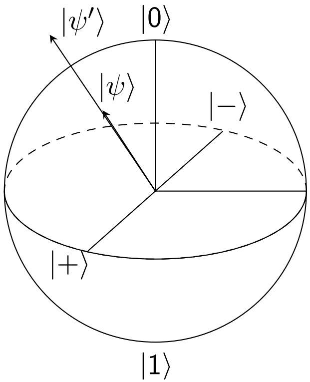
For a pure state, the state space of a qubit can be represented by a **Bloch sphere**.

More generally, define a qu*dit*

$$|\psi\rangle = a_0\,|0\rangle + a_1\,|1\rangle + \cdots + a_{d-1}\,|d-1\rangle$$

$$|\psi\rangle \in \mathcal{H}_d \cong \mathbb{C}^d.$$

Global Phase



Let

$$P_m = P_m^* = P_m P_m$$

be an orthogonal projection corresponding to a measurement outcome m. The probability p_m that outcome m occurs in the state $|\psi\rangle$ is

$$\begin{aligned} p(m) &= \langle\psi| P_m |\psi\rangle \\ &= \langle\psi| e^{-i\theta} P_m e^{i\theta} |\psi\rangle \end{aligned}$$

$$|\psi\rangle = e^{i\theta} |\psi'\rangle \implies |\psi\rangle \sim |\psi'\rangle$$

For this reason, we say that *global phase has no impact on measurement outcomes.*

Quantum Codes

Definition

A quantum code $\mathcal{C}$ is a subspace of a system's state space

$$|\psi_{\text{code}}\rangle \in \mathcal{C} \subseteq \mathcal{H}_d.$$

It can be described through a $d \times d$ projection matrix

$$P_{\mathcal{C}} = P_{\mathcal{C}}^*, \qquad P_{\mathcal{C}} P_{\mathcal{C}} = P_{\mathcal{C}}$$

$$P_{\mathcal{C}} \in M_d(\mathbb{C}) \cong \mathcal{L}(\mathcal{H}_d).$$

We can regard $P_{\mathcal{C}}$ as a boolean asking if a state $|\psi\rangle$ is in the code.

Metric Space

A metric space is a set X possessing a distance function $d : X \times X \to \mathbb{R}$ (in fact, by the following axioms, $d : X \times X \to \mathbb{R}_{\geq 0}$) s.t.

1. $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$.

Hamming Metric

Let $X = \{0, 1\}^n$ be the set of length n bit strings. Define the distance between bit strings $x, y \in X$ as the number of bit values that differ between x and y.

Quantum Metric

A quantum metric on $M_d(\mathbb{C}) \cong \mathcal{L}(\mathcal{H}_d)$ is a nested chain of subspaces

$$I \in \mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \cdots \subseteq M_d(\mathbb{C})$$

invariant under taking Hermitian adjoints and possessing the properties that

1. $\mathcal{V}_0 = \text{span}\{I\}$
2. $\mathcal{V}_t = \mathcal{V}_t^*$
3. $\mathcal{V}_s \mathcal{V}_t \subseteq \mathcal{V}_{s+t}$

These properties are analogous to those in a classical metric space as

1. $\mathcal{V}_0 = \text{span}\{I\} \longleftrightarrow d(x,y) = 0 \iff x = y$
2. $\mathcal{V}_t = \mathcal{V}_t^* \longleftrightarrow d(x,y) = d(y,x)$
3. $\mathcal{V}_s \mathcal{V}_t \subseteq \mathcal{V}_{s+t} \longleftrightarrow d(x,z) \leq d(x,y) + d(y,z)$

Quantum Metric

Each $\mathcal{V}_t$ is defined to be the linear span of operators acting on $\mathcal{H}_d$ which affect at most t qubits. For example, an error $E \in \mathcal{V}_1$ affects one or fewer qubits, but the particular qubit which it affects is all of the qubits in the codeword in quantum superposition.

Quantum Hamming Metric

Up to orthonormal change of basis, a chain of subspaces with bases consisting of the matrices

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

characterize the quantum Hamming metric on a single qubit $M_2(\mathbb{C})$.

Quantum Errors

Definition

An error space $\mathcal{V}_t$ is a self-adjoint subspace of the $d \times d$ matrices with entries in $\mathbb{C}$

$$I \in \mathcal{V}_t = \mathcal{V}_t^* \subseteq M_d(\mathbb{C}) \cong \mathcal{L}(\mathcal{H}_d).$$

$\mathcal{V}_t$ consists of errors E acting on states $|\psi\rangle$ in $\mathcal{C}$.

Example

Take $\mathcal{H} \cong \mathbb{C}[\{0,1\}]^{\otimes 3}$, the quantum Hamming space for three qubits. $X, Y, Z, I \in M_2(\mathbb{C})$, so a 1-qubit error caused by Y looks like

$$E \propto (=_{could}) Y \otimes I \otimes I + I \otimes Y \otimes I + I \otimes I \otimes Y.$$

(A 3-qubit error would have some combination of X, Y, or Z in all positions and I in none.)

Quantum Error Detection Condition

Let

$$E \in \mathcal{V}_t, \quad |\psi\rangle \in \mathcal{C} \subseteq \mathcal{H}_d.$$

If

$$P_{\mathcal{C}} E |\psi\rangle \propto |\psi\rangle \;\; \forall \;\; |\psi\rangle \in \mathcal{C}, E \in \mathcal{V}_t$$

then $\mathcal{C}$ detects errors in $\mathcal{V}_t$.

Let's take a version of this condition more useful for making precise computations.

Define a linear function

$$\epsilon : \mathcal{V}_t \to \mathbb{C}$$

so that

$$P_{\mathcal{C}} E P_{\mathcal{C}} = \epsilon(E) P_{\mathcal{C}} \; \forall \; E \in \mathcal{V}_t.$$

A General Error Operator

Cases for Quantum Errors

- Consider the case of $P_{\mathcal{C}} E P_{\mathcal{C}} = \epsilon(E) P_{\mathcal{C}}$ where $P_{\mathcal{C}} E P_{\mathcal{C}} = 0$. Our successful detection of E is guaranteed. We started with $P_{\mathcal{C}} |\psi\rangle \in \mathcal{C}$ but now we have $E P_{\mathcal{C}} |\psi\rangle \perp \mathcal{C}$.
- But what if $E \propto I$? Then $P_{\mathcal{C}} E |\psi\rangle \propto P_{\mathcal{C}} |\psi\rangle \in \mathcal{C}$ always.

An arbitrary error operator is a linear combination of error which is strictly detectable and error which impacts only global phase.

Consider a strictly detectable error F such that $FP |\psi\rangle \perp \mathcal{C}$ and an inconsequential error $\epsilon(E) I$. Then an arbitrary error E can be written as $E = F + \epsilon(E) I$.

In order to guarantee the detection of E, we have to subtract the portion of $E \propto$ I in the superposition sense.

Slopes as Vector-Valued Eigenvalues

Consider a pre-selected slope ϵ for single error E relative to a one dimensional code containing the state $|\psi\rangle$.

We have

$$\dim \mathcal{C} = 1, \quad P_{\mathcal{C}} = |\psi\rangle\langle\psi|.$$

Notice that

$$\begin{aligned} P_{\mathcal{C}} E P_{\mathcal{C}} &= \epsilon(E) P_{\mathcal{C}} \\ &= |\psi\rangle\langle\psi| E |\psi\rangle\langle\psi| \\ &= \langle\psi| E |\psi\rangle |\psi\rangle\langle\psi| \end{aligned}$$

$$\implies \epsilon(E) = \langle\psi| E |\psi\rangle .$$

Slopes as Vector-Valued Eigenvalues

Suppose we have a d-dimensional code

$$\mathcal{C} = \text{span}\{|\psi_0\rangle, \ldots, |\psi_d\rangle\}$$

with orthonormal basis states $|\psi_0\rangle$ through $|\psi_d\rangle$ and an error space

$$\mathcal{V}_t = \text{span}\{I, E_0, \ldots, E_k\}, \mathcal{V}_t^* = \mathcal{V}_t$$

s.t.

$$|\psi_i\rangle \perp E\,|\psi_j\rangle \;\forall\; E \in \mathcal{V}_t, i \neq j \iff \langle\psi_i|\, E_\ell \,|\psi_j\rangle = \epsilon(E_\ell)\delta_{ij}$$

$$\iff P_{\mathcal{C}} E_\ell P_{\mathcal{C}} = \epsilon(E_\ell) P_{\mathcal{C}}.$$

Slopes as Vector-Valued Eigenvalues

$$\mathcal{C} = \text{span}\{|\psi_0\rangle, \ldots, |\psi_d\rangle\}$$

$$\mathcal{V}_t = \text{span}\{I, E_0, \ldots, E_k\}, \; E = E^* \forall \; E \in \mathcal{V}_t$$

- To each $|\psi_i\rangle, E_j$, there is an associated scalar $\epsilon_i(E_j) = \langle\psi_i| E_j |\psi_i\rangle$.
- To each $|\psi_i\rangle$, there is an associated linear function ϵ_i, with the vector representation

$$\vec{\epsilon}_i(E) = \begin{bmatrix} \langle\psi_i| E_1 |\psi_i\rangle \\ \vdots \\ \langle\psi_i| E_k |\psi_i\rangle \end{bmatrix}.$$

Applying These Ideas

- Seeking a compatible slope betwixt one-dimensional codes to make a code of dimension $\leq d$
- Investigating codes not built on a subspace with commutative error
- Exploring quantum metric spaces with isometry groups possessing nice symmetries
- Using some subset of the above ideas to find bounds on codes in particular families of metric spaces

Thank you!