Connectedness of Varying Markoff Surfaces

Jillian Eddy

UC Davis REU

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Outline

1. Background
   - Markoff triples
   - Involutions
   - Altering the Markoff Equation
2. Varying *a*
   - a = 2
   - a = 4
   - a = 3
     - 2 as a quadratic residue
     - 5 as a quadratic residue
   - a = golden ratio + 2
   - Conclusion

Markoff triples

$$x^2 + y^2 + z^2 = 3xyz$$
$$\downarrow$$
$$x^2 + y^2 + z^2 = xyz$$

$$(x, y, z)$$
$$\downarrow$$
$$(3x, 3y, 3z)$$

Example

$(1, 3, 6)$ as a solution modulo 7:

$x^2 + y^2 + z^2 = xyz$

$1^2 + 3^2 + 6^2 = 1 \cdot 3 \cdot 6$

$46 = 18$

$\equiv 4 \ (mod\ 7)$

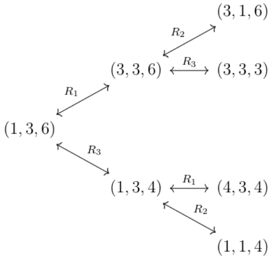
Vieta Involutions

$$R_1(x, y, z) = (yz - x, y, z)$$
$$R_2(x, y, z) = (x, xz - y, z)$$
$$R_3(x, y, z) = (x, y, xy - z)$$

Example

Involution on (1,3,6) modulo 7:
$R_1(1, 3, 6) = (3 \cdot 6 - 1, 3, 6) = (17, 3, 6) \equiv \mathbf{(3,\ 3,\ 6)}\ (mod\ 7)$

Beginning of the involution graph G_7:



Altering the Markoff Equation

$$x^2 + y^2 + z^2 - \mathbf{a} = xyz \ (mod \ p)$$

What if we change a?

- Will $G_{p,a}$ be connected?
- What do disconnected components look like?
- When/where will disconnects occur (depending on our choice of p)?

Caveat: what do we mean by connected?

$x^2 + y^2 + z^2 - \mathbf{5} = xyz \pmod{11}$

```
{(0, 0, 4), (0, 0, 7)}

{(2, 10, 0), (5, 4, 2), (9, 3, 7), (2, 10, 9), (9, 6, 6), (1, 7, 4), (10, 3, 4), (2, 6, 5), (5, 2, 4), (4, 4, 10), (9, 4, 8),
(9, 5, 7), (1, 4, 8), (3, 4, 10), (2, 4, 5), (2, 0, 1), (4, 10, 4), (9, 3, 9), (4, 9, 6), (8, 7, 10), (2, 0, 10), (2, 6, 7), (5,
2, 6), (2, 7, 6), (4, 6, 9), (5, 5, 5), (3, 1, 7), (8, 4, 1), (7, 1, 3), (2, 8, 7), (7, 3, 9), (4, 8, 9), (4, 9, 8), (3, 7, 1),
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6), (9, 0, 1), (4, 7, 5), (9, 0, 10), (1, 4, 7), (4, 3, 10), (6, 7, 7), (6, 9, 4), (7, 4, 5), (7, 2, 8), (9, 2, 10), (10, 4, 4),
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4, 6), (3, 10, 4), (7, 5, 4), (1, 8, 4), (9, 10, 2), (7, 7, 10), (9, 7, 3), (1, 0, 9), (7, 8, 2), (4, 10, 3), (9, 8, 4), (9, 9,
3), (6, 5, 6), (1, 2, 0), (4, 5, 7), (6, 6, 5), (3, 2, 2), (8, 10, 7), (8, 1, 4), (9, 7, 5), (1, 0, 2), (3, 9, 7), (8, 4, 9),
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(4, 2, 3), (6, 6, 9), (10, 7, 7), (2, 3, 2), (1, 9, 0), (4, 1, 7), (6, 2, 5), (8, 2, 7), (1, 9, 9), (7, 6, 2), (9, 6, 4), (5, 6,
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(8, 2, 9)}

{(0, 4, 0), (0, 7, 0)}

{(7, 0, 0), (4, 0, 0)}
```

Figure: "Trivially" disconnected components for p = 11; a = 5

If a is a **quadratic residue** modulo p (i.e. there exists n such that $n^2 \equiv a \ (mod\ p)$) then the two values of n will form inherently disconnected pairs of triples.

Section 2. Varying a

$$x^2 + y^2 + z^2 - \mathbf{a} = xyz \ (mod \ p)$$

a = 2

$$x^2 + y^2 + z^2 - \mathbf{2} = xyz \ (mod \ p)$$

Consider a solution $(1, 1, n)$ to $x^2 + y^2 + z^2 - 2 = xyz \ (mod \ p)$.

$$1^2 + 1^2 + n^2 - 2 = 1 \cdot 1 \cdot n$$
$$n^2 = n$$
$$n^2 - n = 0$$
$$n(n+1) = 0$$
$$n = 0, 1$$

Example

$R_1(1, 1, 0) = (1 \cdot 0 - 1, 1, 0) = (-1, 1, 0)$

$$x^2 + y^2 + z^2 - \mathbf{2} = xyz \ (mod \ p)$$

For solutions (x, y, z), let $x, y, z \in \{0, 1, -1\}$. Without loss of generality, the involution map R_1

$$R_1(x, y, z) = (yz - x, y, z)$$

produces solutions also with values in $\{0, 1, -1\}$.
That is to say, $(yz - x) \in \{0, 1, -1\}$.

Note: $-1 \equiv p - 1 \ (mod \ p)$

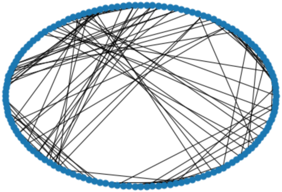


Figure: Graph for p = 13; a = 2

```
{(1, 0, 1), (12, 12, 1), (1, 1, 0), (12, 0, 1), (12, 12, 0), (12, 1, 0), (12, 0, 12), (0, 1, 1), (0, 1, 12), (0, 12, 12), (12,
1, 12), (1, 12, 0), (1, 0, 12), (1, 12, 12), (1, 1, 1), (0, 12, 1)}

{(6, 8, 4), (9, 3, 7), (5, 6, 8), (6, 5, 8), (7, 5, 5), (3, 6, 1), (7, 10, 4), (9, 10, 6), (7, 1, 10), (10, 6, 12), (6, 9, 5),
(5, 9, 0), (8, 9, 7), (8, 0, 4), (10, 7, 4), (5, 0, 9), (5, 5, 5), (9, 5, 0), (3, 12, 7), (0, 8, 4), (7, 12, 3), (8, 6, 4), (4,
8, 0), (8, 9, 0), (7, 3, 9), (6, 1, 3), (4, 0, 5), (5, 8, 6), (7, 8, 8), (10, 4, 7), (5, 5, 7), (4, 7, 10), (7, 4, 10), (1, 6,
3), (10, 3, 1), (5, 8, 8), (5, 0, 4), (1, 10, 3), (9, 0, 8), (1, 3, 6), (7, 10, 1), (6, 12, 10), (10, 6, 9), (8, 8, 5), (10, 12,
6), (10, 3, 3), (6, 3, 1), (10, 7, 1), (12, 7, 3), (4, 7, 5), (3, 3, 10), (6, 4, 8), (4, 8, 6), (7, 4, 5), (8, 7, 9), (9, 6, 1
0), (9, 8, 7), (1, 10, 7), (6, 5, 9), (4, 6, 8), (3, 1, 6), (3, 3, 12), (5, 7, 4), (9, 7, 8), (0, 4, 8), (12, 10, 6), (10, 12, 1
0), (9, 8, 0), (1, 7, 10), (9, 0, 5), (3, 7, 12), (12, 3, 3), (6, 4, 3), (0, 5, 9), (10, 1, 3), (9, 6, 5), (5, 9, 6), (7, 8, 9),
(7, 9, 8), (4, 6, 3), (8, 5, 6), (3, 1, 10), (3, 4, 6), (7, 5, 4), (9, 5, 6), (6, 10, 12), (9, 7, 3), (5, 4, 0), (6, 8, 5), (8,
8, 7), (12, 10, 10), (5, 6, 9), (4, 5, 7), (8, 5, 8), (3, 12, 3), (0, 9, 8), (1, 3, 10), (12, 3, 7), (8, 4, 0), (0, 4, 5), (3,
9, 7), (0, 5, 4), (8, 7, 8), (10, 10, 10), (10, 1, 7), (7, 9, 3), (4, 5, 0), (3, 7, 9), (3, 6, 4), (8, 6, 5), (5, 4, 7), (10, 1
0, 12), (4, 10, 7), (6, 3, 4), (3, 10, 1), (12, 6, 10), (5, 7, 5), (6, 10, 9), (7, 3, 12), (6, 9, 10), (4, 0, 8), (3, 10, 3),
(8, 0, 9), (0, 9, 5), (10, 9, 6), (0, 8, 9), (4, 3, 6), (8, 4, 6)}
```

Figure: Two disconnected components for p = 13; a = 2

a = 4

$$x^2 + y^2 + z^2 - \mathbf{4} = xyz \ (mod \ p)$$

$(2, 2, 2)$ is a solution to the $a = 4$ case for any choice of p:

$$\begin{aligned} x^2 + y^2 + z^2 - 4 &= xyz \\ 2^2 + 2^2 + 2^2 - 4 &= 2 \cdot 2 \cdot 2 \\ 8 &= 8 \end{aligned}$$

Without loss of generality, examine the action of the involution map

$$R_1(x, y, z) = (yz - x, y, z)$$

on this triple:

$$\begin{aligned} R_1(2, 2, 2) &= (2 \cdot 2 - 2, 2, 2) \\ &= (2, 2, 2) \end{aligned}$$

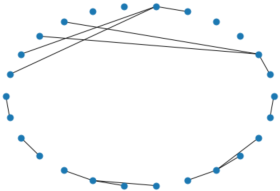


Figure: Disconnected components for p = 5; a = 4

```
{(0, 0, 2), (0, 0, 3)}
{(0, 2, 0), (0, 3, 0)}
{(3, 1, 4), (1, 1, 4), (1, 3, 4), (1, 1, 2)}
{(1, 2, 1), (3, 4, 1), (1, 4, 3), (1, 4, 1)}
{(3, 0, 0), (2, 0, 0)}
{(4, 3, 1), (2, 1, 1), (4, 1, 1), (4, 1, 3)}
{(2, 2, 2)}
{(2, 3, 3)}
{(2, 4, 4), (4, 4, 4), (4, 2, 4), (4, 4, 2)}
{(3, 2, 3)}
{(3, 3, 2)}
```

a = 3

$$x^2 + y^2 + z^2 - \mathbf{3} = xyz \ (mod\ p)$$

Two scenarios in which the solutions will be disconnected:

- 2 is a quadratic residue modulo p
- 5 is a quadratic residue modulo p

2 as a quadratic residue

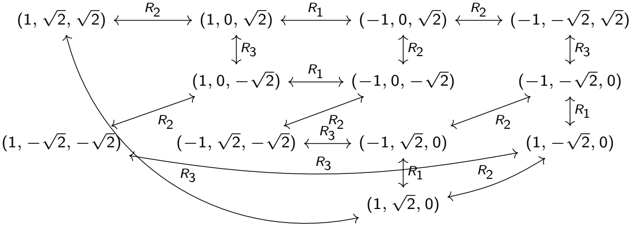
$$x^2 + y^2 + z^2 - \mathbf{3} = xyz \ (mod \ p)$$

Consider a solution $(1, n, n)$,

$$\begin{aligned} 1^2 + n^2 + n^2 - 3 &= 1 \cdot n \cdot n \\ 2n^2 - 2 &= n^2 \\ n^2 &= 2 \\ n &= \pm\sqrt{2} \end{aligned}$$

If there exists some n such that $n^2 \equiv 2 \ (mod \ p)$,
$(1, n, n) = (1, \sqrt{2}, \sqrt{2}) \ (mod \ p)$ will be a valid triple.

A component of triples with entries in $\{0, \pm 1, \pm\sqrt{2}\}$ is formed:



(R_1 involutions which map a triple to itself are omitted for clarity).

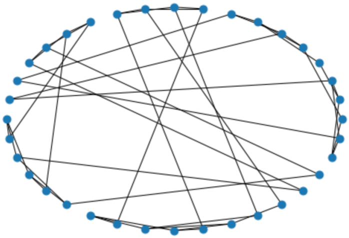


Figure: Disconnected components for p = 7; a = 3

```
{(3, 6, 4), (0, 1, 4), (3, 1, 3), (4, 1, 4), (0, 1, 3), (3, 1, 0), (3, 6, 0), (0, 6, 4), (4, 1, 0), (0, 6, 3), (4, 6, 3), (4, 6,
0)}

{(3, 3, 1), (0, 3, 1), (3, 4, 6), (0, 4, 1), (4, 0, 1), (3, 0, 6), (4, 4, 1), (0, 3, 6), (0, 4, 6), (4, 0, 6), (4, 3, 6), (3, 0,
1)}

{(6, 3, 0), (1, 4, 4), (6, 4, 0), (1, 0, 4), (6, 4, 3), (6, 0, 3), (1, 4, 0), (1, 0, 3), (1, 3, 3), (6, 3, 4), (1, 3, 0), (6, 0,
4)}
```

5 as a quadratic residue

$$x^2 + y^2 + z^2 - \mathbf{3} = xyz \ (mod \ p)$$

Consider a solution $(0, n, n+1)$,

$$\begin{aligned} 0^2 + n^2 + (n+1)^2 - 3 &= 0 \cdot n \cdot (n+1) \\ 2n^2 + 2n - 2 &= 0 \\ n^2 + n - 1 &= 0 \\ n &= \frac{-1 \pm \sqrt{5}}{2} \end{aligned}$$

When 5 is a quadratic residue modulo p,

$$x^2 + y^2 + z^2 - 3 = xyz \ (mod \ p)$$

will produce a disconnected component composed of triples (x, y, z) with values

$$x, y, z \in \{0, \pm 1, \frac{-1 \pm \sqrt{5}}{2}, \frac{-1 \pm \sqrt{5}}{2} + 1\}.$$

5 as a quadratic residue

$\sqrt{5} = 9, 10 \pmod{19}$

$\frac{-1+\sqrt{5}}{2} = \frac{-1+9}{2} = \mathbf{4}$

$\frac{-1-\sqrt{5}}{2} = \frac{-1-9}{2} = -5 \equiv \mathbf{14}$

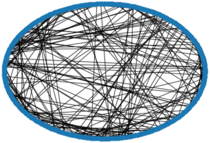


Figure: Disconnected components for p = 19; a = 3

{(15, 5, 0), (4, 14, 18), (0, 4, 14), (0, 15, 14), (1, 5, 1), (15, 14, 0), (1, 5, 4), (4, 5, 0), (1, 15, 14), (15, 5, 18), (5, 0, 4), (18, 4, 14), (15, 1, 14), (1, 4, 5), (1, 18, 4), (4, 14, 0), (4, 1, 5), (14, 18, 4), (14, 1, 18), (14, 18, 1), (14, 1, 15), (18, 5, 15), (15, 18, 18), (5, 4, 1), (0, 14, 15), (18, 5, 18), (0, 15, 5), (1, 15, 1), (14, 0, 4), (0, 5, 15), (18, 4, 1), (1, 1, 15), (1, 14, 18), (4, 18, 14), (1, 14, 15), (5, 18, 15), (18, 1, 14), (5, 18, 18), (15, 1, 1), (18, 18, 15), (5, 15, 0), (5, 0, 15), (15, 18, 5), (14, 15, 1), (15, 0, 5), (5, 15, 18), (5, 1, 1), (1, 1, 5), (5, 4, 0), (15, 14, 1), (5, 1, 4), (4, 18, 1), (18, 15, 18), (18, 1, 4), (15, 0, 14), (4, 0, 5), (0, 14, 4), (4, 0, 14), (14, 0, 15), (4, 5, 1), (18, 18, 5), (0, 4, 5), (18, 14, 1), (14, 4, 0), (18, 14, 4), (14, 15, 0), (1, 4, 18), (18, 15, 5), (1, 18, 14), (4, 1, 18), (14, 4, 18), (0, 5, 4)}

{(0, 7, 7), (7, 12, 8), (15, 12, 15), (6, 7, 15), (7, 4, 13), (6, 0, 9), (1, 11, 8), (11, 18, 11), (16, 5, 9), (11, 3, 1), (14, 5, 16), (5, 5, 3), (7, 15, 6), (12, 0, 7), (13, 3, 11), (4, 7, 15), (3, 6, 10), (8, 16, 1), (9, 5, 16), (14, 3, 9), (11, 6, 12), (18, 11, 16), (6, 11, 12), (10, 14, 16), (16, 9, 6), (10, 16, 13), (5, 3, 5), (6, 7, 8), (8, 8, 8), (13, 10, 16), (14, 14, 3), (3, 14, 14), (7, 4, 15), (16, 14, 5), (16, 11, 18), (10, 3, 6), (7, 8, 6), (12, 12, 11), (11, 12, 6), (7, 13, 11), (9, 14, 3), (8, 13, 16), (4, 4, 12), (9, 0, 13), (4, 12, 4), (1, 3, 11), (5, 16, 14), (11, 8, 11), (13, 10, 0), (13, 8, 12), (8, 18, 3), (15, 4, 15), (11, 13, 3), (12, 11, 12), (8, 7, 12), (6, 9, 16), (16, 13, 8), (10, 10, 14), (11, 1, 8), (7, 13, 4), (12, 7, 8), (9, 0, 6), (7, 11, 13), (14, 10, 10), (18, 11, 11), (4, 12, 6), (6, 11, 16), (11, 6, 16), (6, 15, 7), (6, 9, 0), (13, 0, 10), (3, 14, 9), (9, 3, 13), (16, 13, 10), (0, 12, 12), (13, 9, 3), (6, 3, 8), (3, 10, 5), (10, 9, 5), (5, 9, 16), (18, 8, 8), (4, 15, 7), (3, 1, 11), (15, 15, 4), (13, 4, 7), (8, 11, 11), (16, 10, 14), (10, 6, 0), (7, 7, 11), (13, 11, 3), (6, 8, 3), (13, 8, 16), (0, 6, 10), (16, 11, 6), (11, 7, 7), (11, 1, 3), (13, 15, 12), (12, 4, 4), (7, 6, 15), (6, 3, 10), (7, 8, 12), (15, 13, 12), (16, 18, 11), (9, 14, 9), (9, 16, 6), (3, 18, 8), (12, 8, 13), (10, 5, 3), (7, 12, 0), (9, 10, 5), (1, 11, 3), (0, 13, 9), (1, 8, 16), (11, 13, 7), (3, 8, 18), (1, 16, 8), (3, 5, 10), (0, 12, 7), (7, 6, 8), (10, 3, 5), (12, 4, 6), (10, 13, 16), (18, 8, 3), (6, 16, 9), (11, 8, 1), (11, 11, 18), (14, 10, 16), (15, 7, 4), (5, 10, 3), (8, 12, 7), (8, 6, 3), (10, 16, 14), (8, 1, 16), (13, 11, 7), (10, 13, 0), (13, 9, 0), (4, 4, 4), (4, 13, 7), (13, 12, 8), (6, 16, 11), (7, 0, 12), (12, 6, 11), (16, 1, 8), (15, 7, 6),

a = golden ratio + 2

$$x^2 + y^2 + z^2 - (\tfrac{1 \pm \sqrt{5}}{2} + 2) = xyz \ (mod\ p)$$

Consider $(0, 1, n)$ to be a solution:

$$0^2 + 1^2 + n^2 - (\frac{1+\sqrt{5}}{2} + 2) = 0 \cdot 1 \cdot n$$

$$n^2 = (\frac{1+\sqrt{5}}{2} + 2) - 1$$

$$n = \pm\sqrt{\frac{1+\sqrt{5}}{2} + 1} = \pm\sqrt{\frac{3+\sqrt{5}}{2}} = \pm\sqrt{(\frac{1+\sqrt{5}}{2})^2}$$

$$n = \pm\frac{1+\sqrt{5}}{2}$$

Solutions will produce a disconnected component composed of triples (x, y, z) with values

$$x, y, z \in \{0, \pm 1, \pm(\frac{1 \pm \sqrt{5}}{2})\}.$$

This will occur twice: once for $a = \frac{1+\sqrt{5}}{2} + 2$ and again for $a = \frac{1-\sqrt{5}}{2} + 2$.

$\sqrt{5} = 4, 7 \pmod{11}$

$\frac{1+\sqrt{5}}{2} + 2 = \frac{1+7}{2} + 2 = 4 + 2 = \mathbf{6}$

$\frac{1-\sqrt{5}}{2} + 2 = \frac{1-7}{2} + 2 = -3 + 2 = -1 \equiv \mathbf{10}$

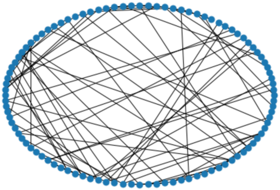


Figure: Disconnected components for p = 11; a = 6

```
{(0, 7, 1), (7, 4, 7), (0, 7, 10), (7, 4, 10), (4, 0, 1), (10, 4, 0), (4, 4, 1), (4, 0, 10), (1, 7, 7), (4, 4, 4), (1, 0, 4),
(0, 4, 1), (7, 10, 4), (7, 1, 7), (1, 0, 7), (0, 4, 10), (1, 7, 0), (4, 10, 7), (10, 7, 4), (1, 4, 4), (10, 0, 7), (7, 7, 4),
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{(0, 6, 5), (1, 6, 3), (6, 0, 6), (5, 6, 8), (8, 3, 1), (6, 8, 10), (8, 5, 1), (5, 5, 0), (3, 10, 5), (3, 1, 8), (5, 5, 3), (6,
5, 8), (1, 3, 6), (10, 8, 6), (3, 6, 1), (8, 10, 6), (8, 1, 3), (3, 8, 1), (1, 8, 5), (0, 5, 6), (5, 1, 8), (5, 3, 5), (8, 6,
5), (5, 8, 1), (3, 5, 5), (6, 0, 5), (10, 3, 3), (6, 3, 1), (6, 6, 0), (6, 6, 3), (8, 5, 6), (5, 0, 6), (1, 3, 8), (10, 8, 8),
(3, 3, 10), (8, 10, 8), (3, 6, 6), (10, 5, 3), (8, 1, 5), (0, 5, 5), (10, 6, 8), (5, 6, 0), (8, 6, 10), (0, 6, 6), (8, 8, 10),
(5, 3, 10), (3, 5, 10), (6, 1, 3), (6, 10, 8), (6, 8, 5), (6, 3, 6), (3, 10, 3), (10, 3, 5), (5, 0, 5), (1, 5, 8), (5, 8, 6),
(6, 5, 0), (3, 1, 6), (1, 8, 3), (5, 10, 3)}
```

Conclusive example: p = 31

- a = 2
  Always disconnected no matter the choice of *p*
- a = 4
  Always disconnected no matter the choice of *p*
- a = 3
  2 is a quadratic residue modulo 31 ($8^2, 23^2 \equiv 2 \ (mod\ 31)$)
  5 is a quadratic residue modulo 31 ($6^2, 25^2 \equiv 5 \ (mod\ 31)$)
- a = golden ratio + 2
  5 is a quadratic residue modulo 31,
  $a = \frac{1+\sqrt{5}}{2} + 2 = \frac{1+25}{2} + 2 = 15$
  $a = \frac{1-\sqrt{5}}{2} + 2 = \frac{1-25}{2} + 2 = 21$

Conclusion

Theorem

The involution graph $G_{p,a}$ of integer solutions modulo p to the equation

$$x^2 + y^2 + z^2 - a = xyz$$

is nontrivially disconnected for values of a

- a $=2$
- a $=4$
- a $=3$
- a $=$ "golden ratio" $+ 2$

and connected for all other values of a.

Thank You!

Program directors Greg Kuperberg and Javier Arsuaga
Advisors Elena Fuchs and Daniel Martin
Groupmates Nico Tripeny and Devin Vanyo