Solving Exact Cover Problems

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UC Davis

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Tiling Problems

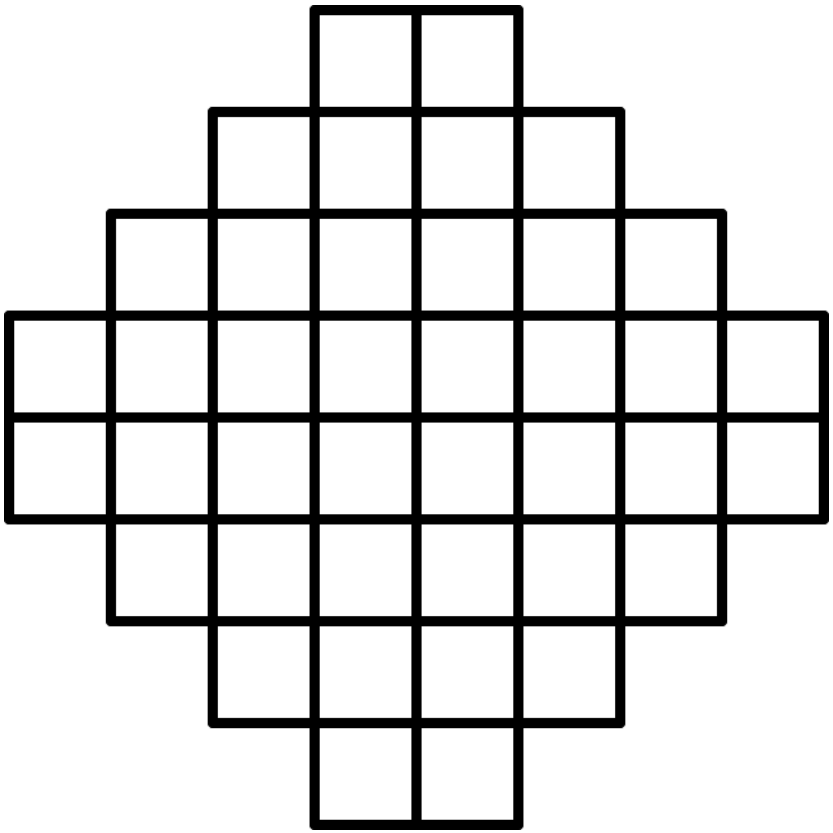

Figure: Aztec Diamond of Order 4

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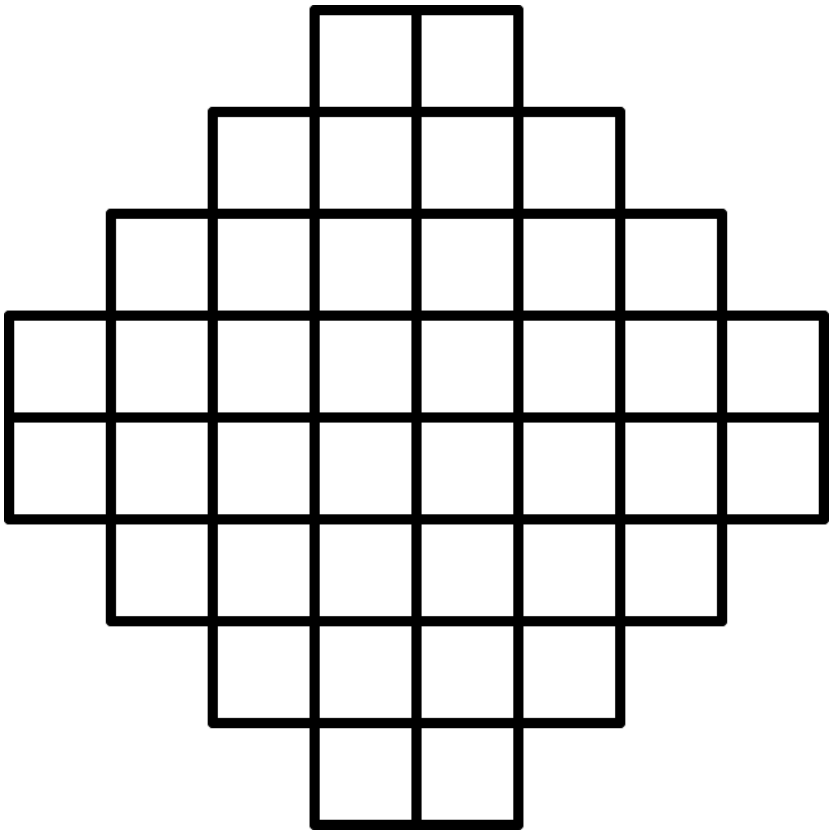

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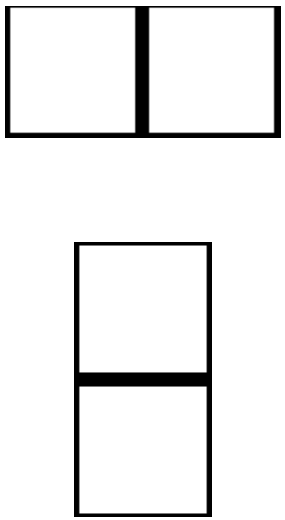

Figure: Tile Types

Set Covers

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- Consider a family of subsets $S \subseteq U$.
- Given (U, S), a set cover is a subfamily $S' \subseteq S$ such that $\bigcup_{s' \in S'} s = U$.

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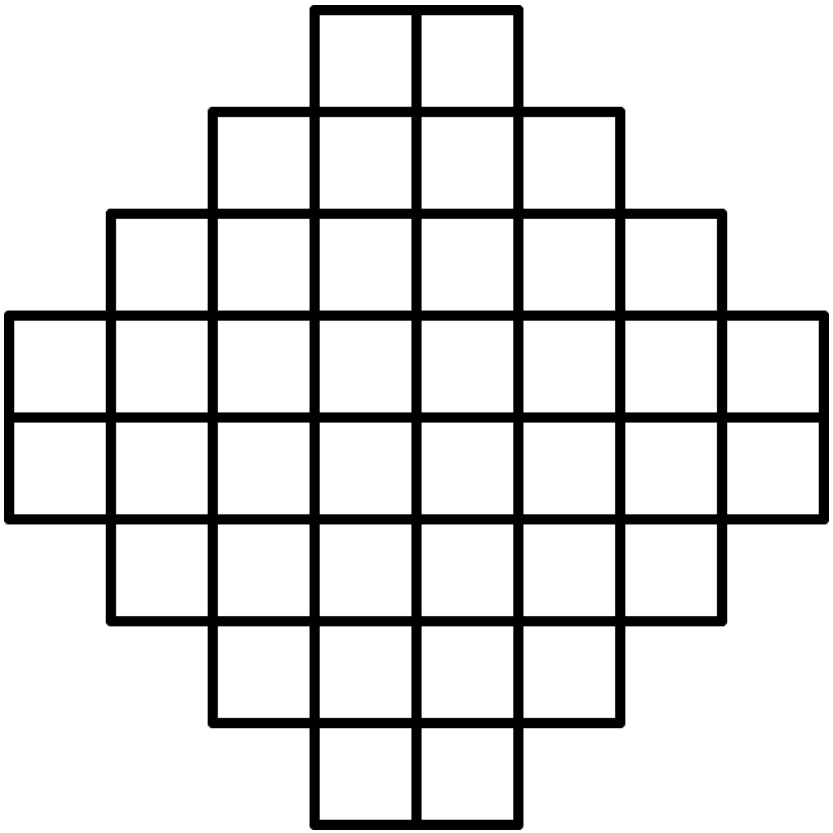
- An exact set cover (or simply exact cover) is a pairwise disjoint set cover.
- Here, we are interested in determining how many exact covers there are of a given (U, S).

Tiling Problems As Exact Cover Problems



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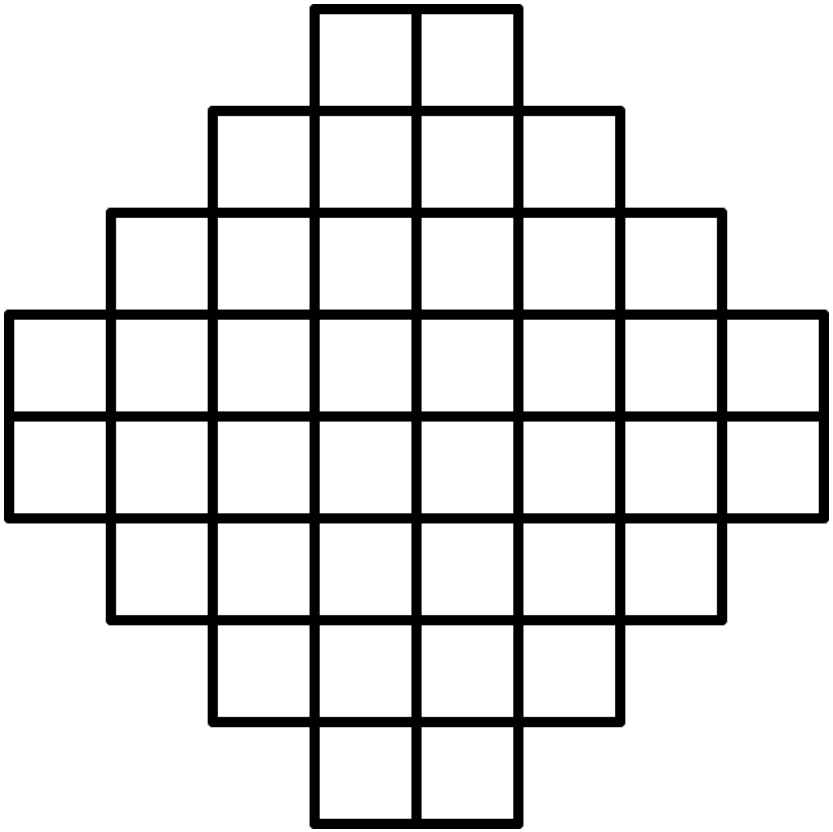

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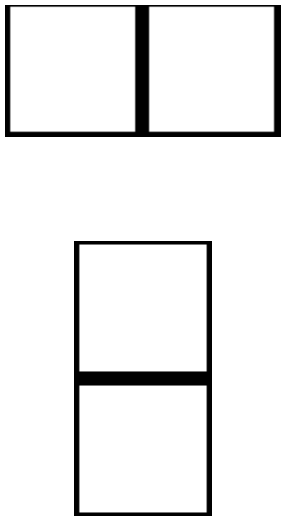

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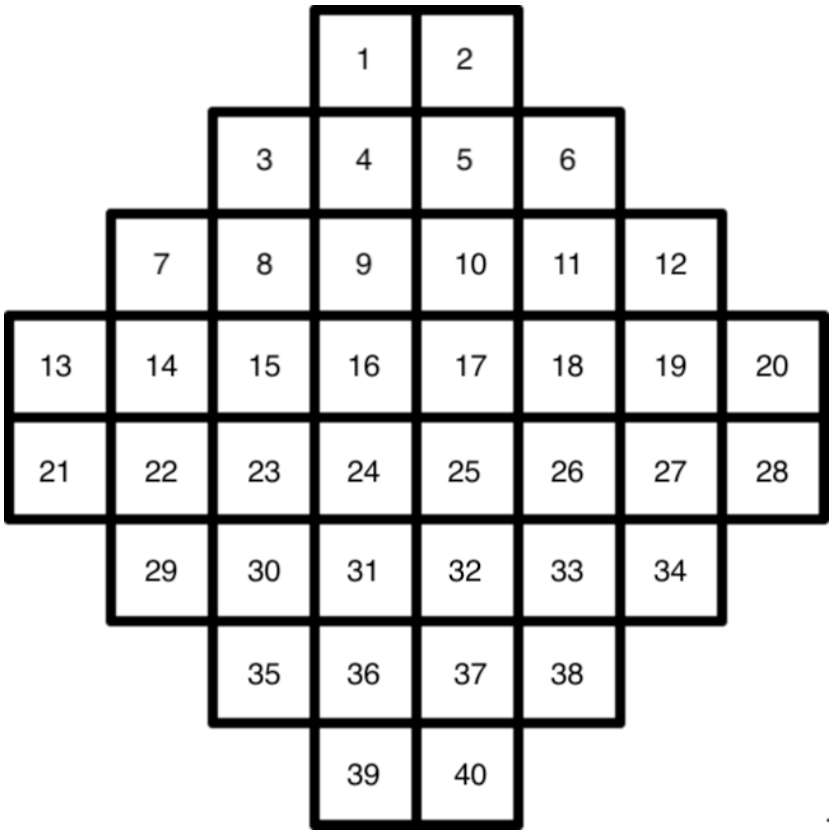

Figure: Enumerated Region

$$\text{Tiles} = \{(1,2),(1,4),(2,5),(3,4),...\}$$

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- The set of all numbers used to enumerate the region is our universe U and the set of tiles is our family S.
- The question of finding an exact cover then becomes isomorphic to the question of how to perfectly tile our region with non-overlapping tiles.
- So, given a region that we can enumerate and tile types that we can use to find tiles, solving the exact cover problem that arises will tell us exactly how many perfect tilings of that region are possible with those given tile types.

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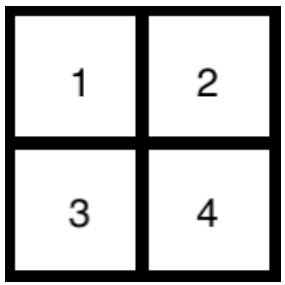

Figure: Aztec Diamond of Order 1

- Suppose we are tiling the above region with the tiles $(1, 2)$ and $(3, 4)$. We don't want $(1, 2) \cup (3, 4)$ and $(3, 4) \cup (1, 2)$ to count as two different perfect tilings since they are equal to the same cover set: $\{(1, 2), (3, 4)\}$

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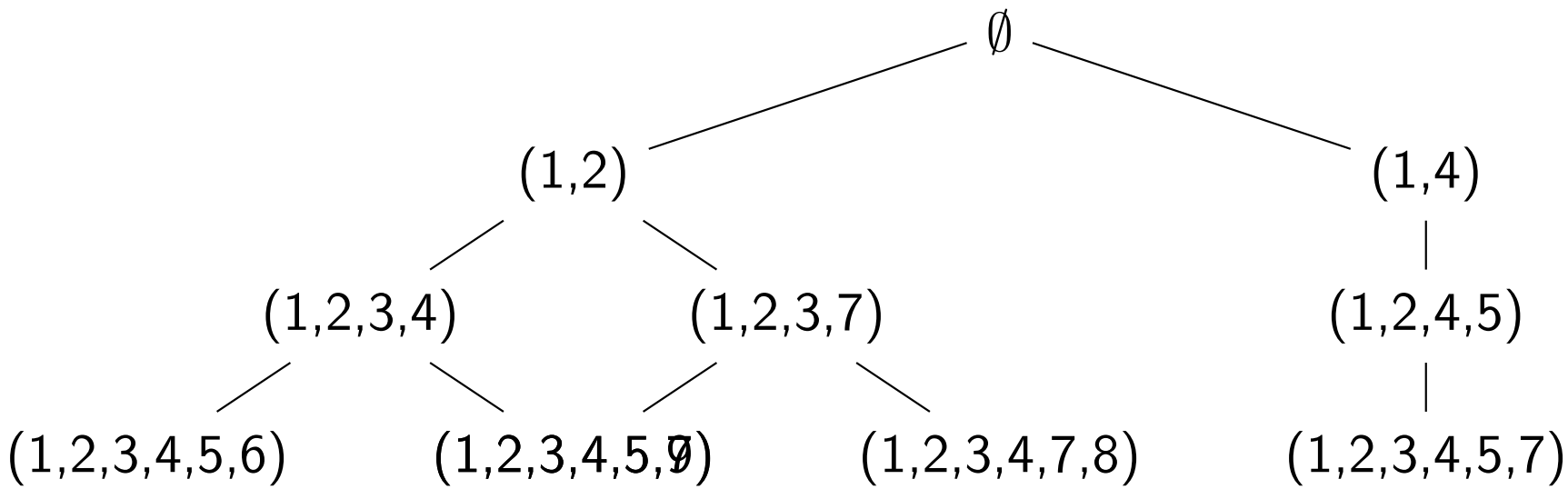
- With this system, we would have required $(1, 2)$ to be placed before $(3, 4)$.

The Naïve Approach



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- The main problem is that this blows up really quickly. In fact, this method has a time cost that is exponential in the area of the region being tiled.
- Solving exact cover problems is always going to be exponential, but we can do better than this approach with a method called dynamic programming.

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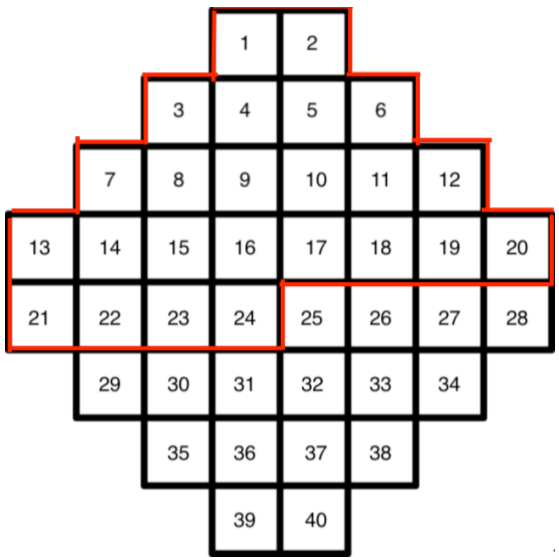

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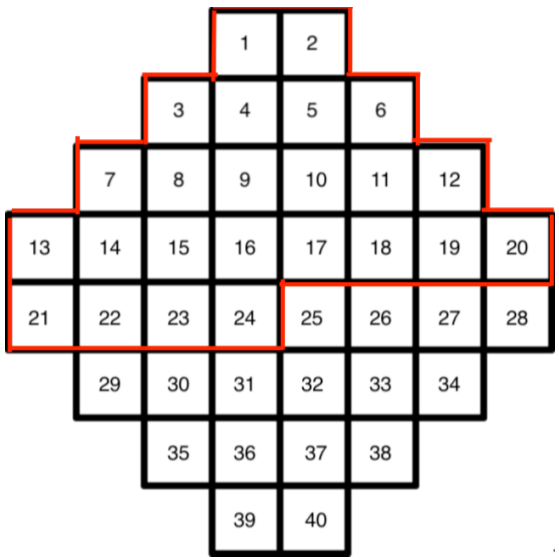

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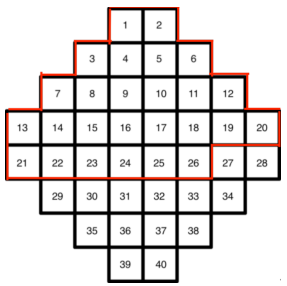

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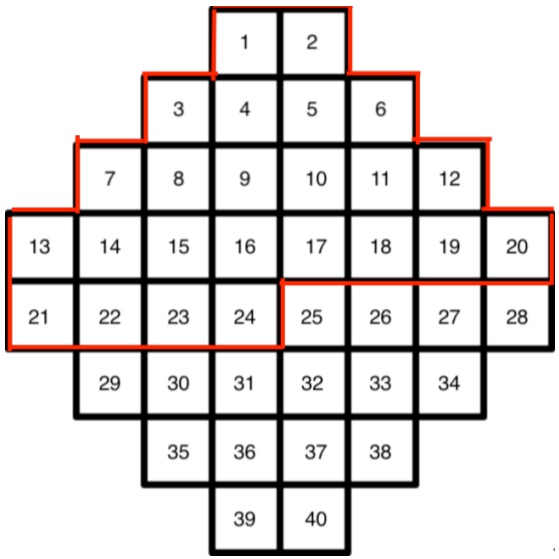

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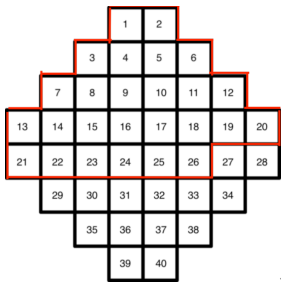

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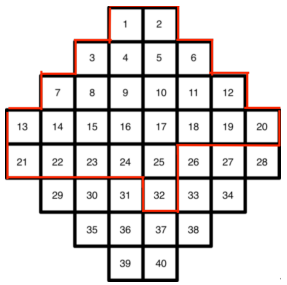

Figure: Option 2

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- Of course, there is a space trade-off, so this is supposing that time is more valuable than space.
- This will be central to my program.

Motivation

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- David's code is maximally efficient, but there are a couple of problems:
  1. It is not generalized to solve exact cover problems in general.
  2. It can be made to be more user-friendly. (e.g. It doesn't have a cover script.)

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- The idea was to write out the program in Python and to swap out the computationally intensive part (the dynamic programming) for David's generalized exact cover solver that he would write in $C++$ for efficiency.
- The program could then be assembled in Cython.

Step 0: User Input

- The ECPS takes in input from the user in the form of ASCII art of regions and tile types with hashtags for tileable units and anything else for non-tileable units.

```
...##...
..####..
.######.
########
########
.######.
..####..
...##...

##

#
#
```

Figure: Input For The ECPS

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- This list of lists provides a coordinate system whereby a symbol in the region is at (x, y) if it is x down steps and y steps right from the top left symbol.
- The region gets sent to a function that assigns a number to each of the hashtags, starting from 1 and not skipping numbers, and returns a coordinate dictionary, or cdict, whose keys are coordinates that contain hashtags and whose values are the numbers associated with those hashtags.

Step 1: Enumeration

- For example, numbering up-to-down, left-to-right, numbering the region from step 0 would return to $\{(0,4):1,(0,5):2,(1,3):3,(1,4):4,(1,5):5,(1,6):6,\ldots\}$.

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- This gives us our universe for the exact cover problem: the set of values of this cdict.
- The decision to go from up-to-down, left-to-right works for this particular region, but is not necessarily the best option for regions in general. Right now, the code only numbers in this way, but I aim to fix that. I may address this at the end if there is time.

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- Since the numbering started from the top left for both the region and the tile types, there will be a tile in the top left of the region iff each (x_k, y_k) in the region is a hashtag.
- In fact, we can find all possible tiles by shifting the tile type coordinates down and to the right together and checking whether they are all positions of hashtags in the region.

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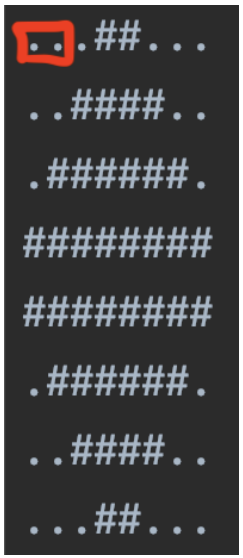

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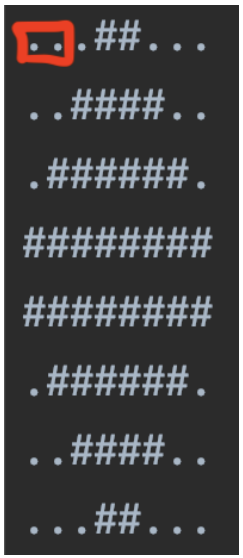

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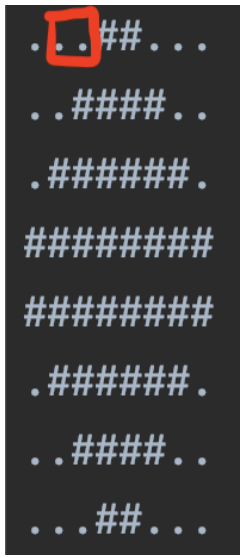

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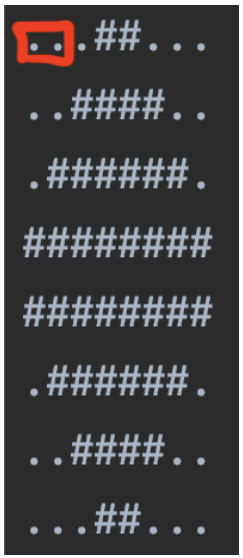

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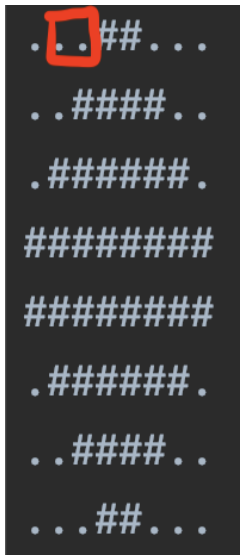

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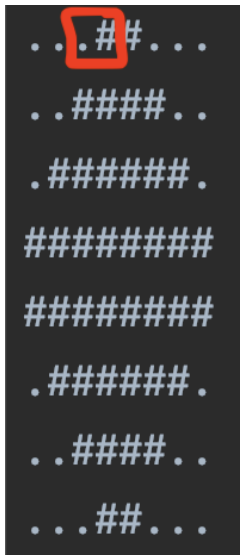

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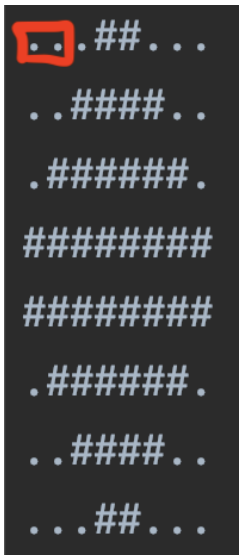

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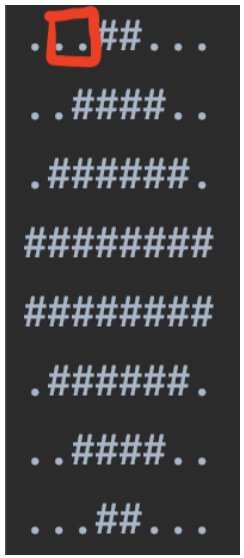

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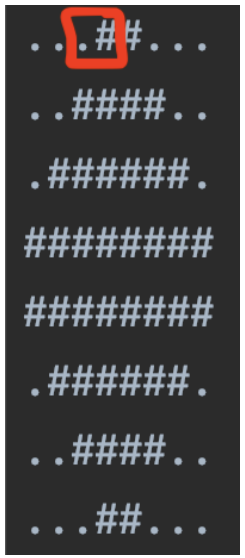

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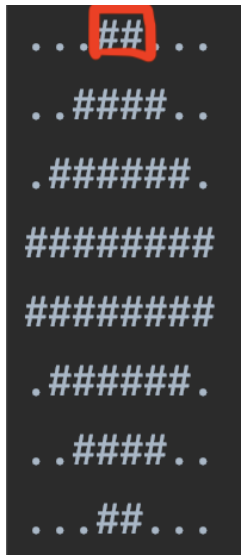

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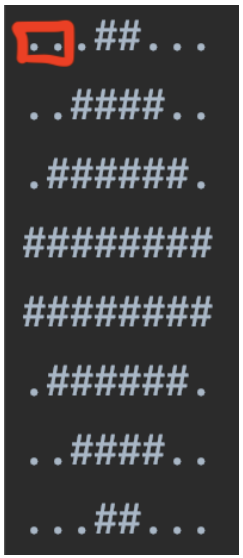

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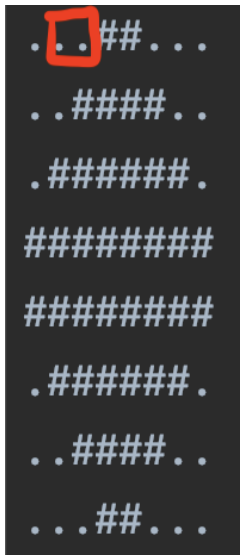

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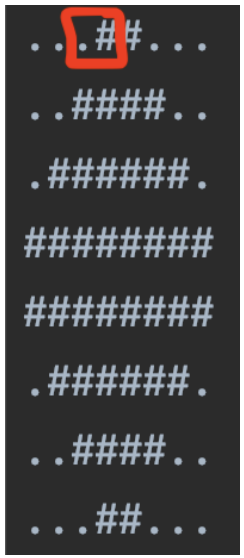

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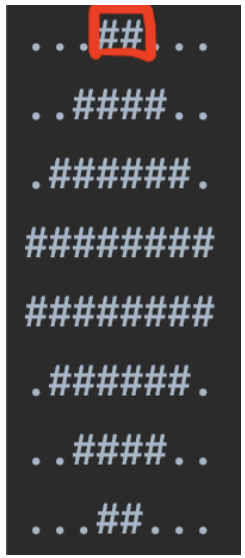

Figure: Tile

- If a potential tile passes the test, the tile is saved as a tuple containing the numbers associated with the hashtags it covers. So, the tile above would be $(1, 2)$.

Solving the Cover set problem

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- From this point on, a "tile" is just another word for a subset in the family S and "tiles" is another word for S itself. We retain the terminology for consistency with the first part of the ECPS, but this part can, importantly, solve exact cover problems in general, not just tiling problems.

Step 2.5: Sorting

- Each of our tiles is a now in the form of a tuple that contains the numbers it covers. We can represent the same information in a binary sequence of length n, where n is the dimension of U and the i'th position from the left is a 1 iff that tile covers i.

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- The ECPS creates a list of all key reps of tiles, called a key chain.

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- When we start having partial tilings, the keys of the i'th dictionary in counts will be the key reps for the the partial tilings whose first 0 from the left is in the i'th position and its values will be how many times the partial tiling has been achieved. The integer keeps track of perfect tilings.

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- At first, our only partial tiling is the trivial one, represented by 0, of which there is 1.

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- if the tile is compatible with the partial tiling, sum them to get a new partial tiling.
- Update the value of the new partial tiling.
  - If it is a full tiling, update counts[0].
  - If it has a hole, send it to the appropriate dict in counts as a key with the updated value.

Future Work

1. Write a module that will determine the most efficient numbering of a region.
2. David has not gotten back to Greg about the code for the computationally intensive part, so I plan to write that up in Cython myself.

Thank You!

- Thank you to Greg for you guidance!
- Thank you to David for your code!
- Thank you to all of you for listening!